DISCRETE PAINLEVÉ EQUATIONS FOR A CLASS OF P_{VI} τ-FUNCTIONS GIVEN AS $U(N)$ AVERAGES

P.J. FORRESTER AND N.S. WITTE

ABSTRACT. In a recent work difference equations (Laguerre-Freud equations) for the bi-orthogonal polynomials and related quantities corresponding to the weight on the unit circle $w(z)=\prod_{j=1}^{m}(z-z_j(t))^{\rho_j}$ were derived.Here it is shown that in the case $m=3$ these difference equations, when applied to the calculation of the underlying $U(N)$ average, reduce to a coupled system identifiable with that obtained by Adler and van Moerbeke using methods of the Toeplitz lattice and Virasoro constraints. Moreover it is shown that this coupled system can be reduced to yield the discrete fifth Painlevé equation dP_{V} as it occurs in the theory of the sixth Painlevé system. Methods based on affine Weyl group symmetries of Bäcklund transformations have previously yielded the dP_{V} equation but with different parameters for the same problem. We find the explicit mapping between the two forms. Applications of our results are made to give recurrences for the gap probabilities and moments in the circular unitary ensemble of random matrices, and to the diagonal spin-spin correlation function of the square lattice Ising model.

1. Introduction

In a recent work [12] we undertook a study into differential and difference structures associated with bi-orthogonal polynomials for the weight on the unit circle

$$w(z)=\prod_{j=1}^{m}(z-z_j(t))^{\rho_j},\quad z=e^{i\theta},\quad \theta\in(-\pi,\pi]. \tag{1.1}$$

Our motivation was to eventually use these results to provide recurrences in N for the random matrix average

$$\Big\langle\prod_{l=1}^{N}w(z_l)\Big\rangle_{U(N)}:=\frac{1}{(2\pi)^N N!}\int_{-\pi}^{\pi}d\theta_1\dots\int_{-\pi}^{\pi}d\theta_N\prod_{l=1}^{N}w(z_l)\prod_{1\le j<k\le N}|z_k-z_j|^2. \tag{1.2}$$

Here $U(N)$ denotes the unitary group of $N\times N$ matrices, and the right hand side exhibits the corresponding eigenvalue probability density function in the case of the Haar (uniform) measure.

In this work we fulfill our original goal in the case $m=3$ by specialising the results of [12] as relevant to the calculation of (1.2) with the piecewise continuous

2000 *Mathematics Subject Classification.* 33C45, 33E17, 15A52, 82B23.

version of (1.1) in the case $m = 3$,

$$w(z) = t^{-\mu} z^{-\mu-\omega} (1+z)^{2\omega_1} (1+tz)^{2\mu} \begin{cases} 1 & \theta \notin (\pi-\phi, \pi) \\ 1-\xi & \theta \in (\pi-\phi, \pi) \end{cases}, \tag{1.3}$$

where μ, ω_1, ω_2 are complex parameters ($\omega = \omega_1 + i\omega_2$) and $\xi, t = e^{i\phi}$ are complex variables ($\phi \in [0, 2\pi)$). By so studying (1.3) we are also completing a study begun in [13] on recurrences satisfied by $U(N)$ averages in random matrix theory from the viewpoint of orthogonal polynomial theory.

The average (1.2) with weight (1.3) occurs in a variety of problems from mathematical physics. The identification of this average as a τ-function for the Painlevé VI system [11] has led to its characterisation in terms of a solution of the σ-form of the Painlevé VI equation. Regarding specific examples, we first mention that the case $(\omega_1, \omega_2, \mu) = (0, 0, 0)$ is the generating function for the probability that the interval $(\pi - \phi, \pi)$ contains exactly k eigenvalues in Dyson's circular unitary ensemble CUE_N (which is equivalent to the unitary group with Haar measure). Similarly the case $(\omega_1, \omega_2, \mu) = (1, 0, 1)$ is (apart from a simple factor) the generating function for the probability density function of the event that two eigenvalues from CUE_{N+2} are an angle ϕ apart with exactly k eigenvalues in between. The case $\xi = 2$, $\omega_2 = 0$, $\mu = \omega_1 = 1/2$ corresponds to the density matrix for the impenetrable Bose gas in periodic boundary conditions. It was studied in detail in [9]. Furthermore, in the case $\xi = 0$, one sees that (1.3) includes as special cases

$$\Big\langle \prod_{l=1}^{N} z_l^{1/4} |1+z_l|^{-1/2} (1+k^{-2} z_l)^{1/2} \Big\rangle_{U(N)}, \qquad 1/k^2 \le 1, \tag{1.4}$$

$$\Big\langle \prod_{l=1}^{N} (1+1/z_l)^{v'} (1+q^2 z_l)^{v} \Big\rangle_{U(N)}, \qquad q^2 < 1. \tag{1.5}$$

The average (1.4) is equivalent to the Toeplitz determinant given by Onsager for the diagonal spin-spin correlation in the two-dimensional Ising model [21]. As a PVI τ-function it has been studied in [15] and [11]. The average (1.5) occurs as a cumulative probability density in the study of processes relating to increasing subsequences [2, 5].

In the case $\xi = 0$ of (1.3), the same problem as we are addressing here has been previously considered by Adler and van Moerbeke [1], using methods of the Toeplitz lattice and Virasoro constraints. The difference equations obtained there were not identified with known integrable difference equations. Here we find that our formalism of bi-orthogonal polynomials leads to the very same difference equations. Moreover, we are able to show that they can be reduced to examples of the discrete

Painlevé V equation $\mathrm{dP_V}$,

$$g_{n+1}g_n = t\frac{(f_n+1-\alpha_2)(f_n+1-\alpha_0-\alpha_2)}{f_n(f_n+\alpha_3)}, \tag{1.6}$$

$$f_n + f_{n-1} = -\alpha_3 + \frac{\alpha_1}{g_n-1} + \frac{\alpha_4 t}{g_n-t}, \tag{1.7}$$

where $\alpha_1 \mapsto \alpha_1+1, \alpha_2 \mapsto \alpha_2-1, \alpha_4 \mapsto \alpha_4+1$ as $n \mapsto n+1$, fundamental in the theory of the $\mathrm{P_{VI}}$ system for its relationship to the Bäcklund transformations [11], and its association with the degeneration of the rational surface $D_4^{(1)} \to D_5^{(1)}$ in the space of initial conditions [23].

The $\mathrm{dP_V}$ system in relation to $U(N)$ averages (specifically (1.5)) was first found in the work of Borodin [6],[7]. Subsequently the present authors [10] used methods based on the affine Weyl group symmetries of Bäcklund transformations for the $\mathrm{P_{VI}}$ system to give $\mathrm{dP_V}$ recurrences for (1.2) with weight (1.3). These are different to the form of $\mathrm{dP_V}$ found from our transformation of the recurrences from the bi-orthogonal polynomial theory, thus raising the question as to the relationship between the two. This we answer by providing the explicit transformation formulae.

A practical side of our work is that the reduced (in order) form of the Adler and van Moerbeke type recurrences are typically much simpler than their equivalent form expressed in terms of $\mathrm{dP_V}$. We make explicit these recurrences in the cases of the characteristic polynomial and gap probability for the CUE_N, and the case of the diagonal spin-spin correlation function for the square lattice Ising model.

In Section 2 general formulae from [12] required in subsequent sections are recalled. These formulae are used in Section 3 to derive N-recurrences in the case of weight (1.3), and furthermore a transformation to the $\mathrm{dP_V}$ system (1.6), (1.7) is given. The explicit transformation between the latter, and the $\mathrm{dP_V}$ system found in relation to (1.3) as a consequence of the Okamoto τ-function theory [10], is established in Section 4. Application of the recurrences to random matrices and the Ising model is given in Section 5.

2. Bi-orthogonal Polynomial Formalism

From the viewpoint of our work [12], the weight (1.3) is a particular example of the regular semi-classical class (1.1), characterised by a special structure of their logarithmic derivatives

$$\frac{1}{w(z)}\frac{d}{dz}w(z) = \frac{2V(z)}{W(z)} = \sum_{j=1}^{m}\frac{\rho_j}{z-z_j}, \quad \rho_j \in \mathbb{C}. \tag{2.1}$$

Here $V(z)$, $W(z)$ are polynomials with $\deg V(z) < m, \deg W(z) = m$. We define bi-orthogonal polynomials $\{\phi_n(z), \bar{\phi}_n(z)\}_{n=0}^{\infty}$ with respect to the weight $w(z)$ on the unit circle by the orthogonality relation

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\phi_m(\zeta)\bar{\phi}_n(\bar{\zeta}) = \delta_{m,n}, \tag{2.2}$$

where $\mathbb{T}$ denotes the unit circle $|\zeta| = 1$ with $\zeta = e^{i\theta}$, $\theta \in (-\pi, \pi]$. Notwithstanding the notation, $\bar{\phi}_n$ is not in general equal to the complex conjugate of ϕ_n. We set

$$\phi_n(z) = \kappa_n z^n + l_n z^{n-1} + m_n z^{n-2} + \ldots + \phi_n(0),$$
$$\bar{\phi}_n(z) = \kappa_n z^n + \bar{l}_n z^{n-1} + \bar{m}_n z^{n-2} + \ldots + \bar{\phi}_n(0),$$

where again $\bar{l}_n$, $\bar{m}_n$, $\bar{\phi}_n(0)$ are not in general equal to the corresponding complex conjugate. The coefficients are related by many coupled equations, two of the simplest being

$$\kappa_n^2 = \kappa_{n-1}^2 + \phi_n(0)\bar{\phi}_n(0), \quad \frac{l_n}{\kappa_n} - \frac{l_{n-1}}{\kappa_{n-1}} = r_n \bar{r}_{n-1}. \tag{2.3}$$

Denote the $U(N)$ average (1.2) by $I_N[w]$. It is a basic fact that $I_N[w]$ can also be written as the Toeplitz determinant

$$I_n[w] = \det \left[\int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} w(\zeta) \zeta^{-j+k} \right]_{0 \le j,k \le n-1}. \tag{2.4}$$

With the so-called reflection coefficients specified by

$$r_n = \frac{\phi_n(0)}{\kappa_n}, \quad \bar{r}_n = \frac{\bar{\phi}_n(0)}{\kappa_n}, \tag{2.5}$$

it is a well known result in the theory of Toeplitz determinants that

$$\frac{I_{n+1}[w] I_{n-1}[w]}{(I_n[w])^2} = 1 - r_n \bar{r}_n. \tag{2.6}$$

Introduce the reciprocal polynomial $\phi_n^*(z)$ of the nth degree polynomial $\bar{\phi}_n(z)$ by

$$\phi_n^*(z) := z^n \bar{\phi}_n(1/z). \tag{2.7}$$

Fundamental to our study [12] is the matrix

$$Y_n(z;t) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z)/w(z) \\ \phi_n^*(z) & -\epsilon_n^*(z)/w(z) \end{pmatrix}, \tag{2.8}$$

where

$$\epsilon_n(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) \phi_n(\zeta), \tag{2.9}$$

$$\epsilon_n^*(z) := -z^n \int_{\mathbb{T}} \frac{d\zeta}{2\pi i \zeta} \frac{\zeta + z}{\zeta - z} w(\zeta) \bar{\phi}_n(\bar{\zeta}). \tag{2.10}$$

Thus we obtained the difference system

$$Y_{n+1} := K_n Y_n = \frac{1}{\kappa_n} \begin{pmatrix} \kappa_{n+1} z & \phi_{n+1}(0) \\ \bar{\phi}_{n+1}(0) z & \kappa_{n+1} \end{pmatrix} Y_n, \tag{2.11}$$

and the differential system

$$
\begin{aligned}
(2.12)\quad & \frac{d}{dz}Y_n := A_nY_n \\
&= \frac{1}{W(z)}\begin{pmatrix} -\left[\Omega_n(z)+V(z)-\dfrac{\kappa_{n+1}}{\kappa_n}z\Theta_n(z)\right] & \dfrac{\phi_{n+1}(0)}{\kappa_n}\Theta_n(z) \\ -\dfrac{\bar{\phi}_{n+1}(0)}{\kappa_n}z\Theta_n^*(z) & \Omega_n^*(z)-V(z)-\dfrac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z) \end{pmatrix} Y_n.
\end{aligned}
$$

For regular semi-classical weights (1.1), the functions $\Theta_n(z)$, $\Theta_n^*(z)$, $\Omega_n(z)$ and $\Omega_n^*(z)$ in (2.12) are polynomials of degree $\deg\Omega_n(z) = \deg\Omega_n^*(z) = m-1$, $\deg\Theta_n(z) = \deg\Theta_n^*(z) = m-2$, independent of n. Explicitly, for large z

$$
\begin{aligned}
(2.13)\quad \Theta_n(z) &= (n+1+\sum_{j=1}^m \rho_j)\frac{\kappa_n}{\kappa_{n+1}}z^{m-2} \\
&+ \Bigg\{ -[(n+1+\sum_{j=1}^m\rho_j)\sum_{j=1}^m z_j - \sum_{j=1}^m \rho_j z_j]\frac{\kappa_n}{\kappa_{n+1}} + (n+2+\sum_{j=1}^m\rho_j)\frac{\kappa_n^3}{\kappa_{n+1}^2\kappa_{n+2}}\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} \\
&\qquad - (n+\sum_{j=1}^m\rho_j)\frac{\phi_{n+1}(0)\bar{\phi}_n(0)}{\kappa_{n+1}\kappa_n} - 2\frac{\kappa_n l_{n+1}}{\kappa_{n+1}^2}\Bigg\} z^{m-3} + \mathrm{O}(z^{m-4}),
\end{aligned}
$$

$$
\begin{aligned}
(2.14)\quad \Omega_n(z) &= (1+{}^1\!/\!_2\sum_{j=1}^m\rho_j)z^{m-1} \\
&+ \Bigg\{ -{}^1\!/\!_2(\sum_{j=1}^m\rho_j)(\sum_{j=1}^m z_j) + {}^1\!/\!_2\sum_{j=1}^m \rho_j z_j - \sum_{j=1}^m z_j \\
&\qquad + (n+2+\sum_{j=1}^m\rho_j)\frac{\kappa_n^2}{\kappa_{n+2}\kappa_{n+1}}\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} - \frac{l_{n+1}}{\kappa_{n+1}}\Bigg\} z^{m-2} + \mathrm{O}(z^{m-3}).
\end{aligned}
$$

Also obtained in [12], and relevant to the present study, are the small z expansions

$$
\begin{aligned}
(2.15)\quad \Theta_n(z) &= [2V(0)-nW'(0)]\frac{\phi_n(0)}{\phi_{n+1}(0)} \\
&+ \Bigg\{ [2V'(0)-\tfrac{1}{2}nW''(0)]\frac{\phi_n(0)}{\phi_{n+1}(0)} + [2V(0)-(n-1)W'(0)]\frac{\kappa_n\phi_{n-1}(0)}{\kappa_{n-1}\phi_{n+1}(0)} \\
&+ \left([(n+1)W'(0)-2V(0)]\frac{\bar{l}_{n+1}}{\kappa_{n+1}} - [(n-1)W'(0)-2V(0)]\frac{\bar{l}_{n-1}}{\kappa_{n+1}}\right)\frac{\phi_n(0)}{\phi_{n+1}(0)}\Bigg\} z \\
&+ \mathrm{O}(z^2),
\end{aligned}
$$

$$
\begin{aligned}
(2.16)\quad \Omega_n(z) &= V(0)-nW'(0) \\
&+ \Bigg\{ V'(0)-\tfrac{1}{2}nW''(0) + \left(V(0)\frac{\kappa_n}{\kappa_{n+1}} + [V(0)-nW'(0)]\frac{\kappa_{n+1}}{\kappa_n}\right)\frac{\phi_n(0)}{\phi_{n+1}(0)} \\
&\qquad + [V(0)-nW'(0)]\frac{\bar{l}_n}{\kappa_n} - [V(0)-(n+1)W'(0)]\frac{\bar{l}_{n+1}}{\kappa_{n+1}}\Bigg\} z + \mathrm{O}(z^2).
\end{aligned}
$$

Analogous formulae hold for $\Theta^*_n(z)$ and $\Omega^*_n(z)$ (these can be found in [12]).

The coefficient functions $\Omega_n(z), \Omega^*_n(z), \Theta_n(z), \Theta^*_n(z)$ satisfy sets of difference and functional relations given in [12]. Relevant for the present study are the coupled equations

$$\begin{aligned}\left(\frac{\phi_{n+1}(0)}{\phi_n(0)}+\frac{\kappa_{n+1}}{\kappa_n}z\right)(\Omega_{n-1}(z)-\Omega_n(z))\\ +\frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z\Theta_{n+1}(z)-\frac{\kappa_{n-1}\phi_{n+1}(0)}{\kappa_n\phi_n(0)}z\Theta_{n-1}(z)-\frac{\phi_{n+1}(0)}{\phi_n(0)}\frac{W(z)}{z}=0,\end{aligned} \tag{2.17}$$

$$\begin{aligned}\left(\frac{\kappa_{n+1}}{\kappa_n}+\frac{\bar\phi_{n+1}(0)}{\bar\phi_n(0)}z\right)(\Omega^*_{n-1}(z)-\Omega^*_n(z))\\ +\frac{\kappa_n\bar\phi_{n+2}(0)}{\kappa_{n+1}\bar\phi_{n+1}(0)}z\Theta^*_{n+1}(z)-\frac{\kappa_{n-1}\bar\phi_{n+1}(0)}{\kappa_n\bar\phi_n(0)}z\Theta^*_{n-1}(z)+\frac{\kappa_{n+1}}{\kappa_n}\frac{W(z)}{z}=0,\end{aligned} \tag{2.18}$$

$$\begin{aligned}\Omega^*_{n+1}(z)+\Omega_n(z)-\left(\frac{\kappa_{n+2}}{\kappa_{n+1}}+\frac{\bar\phi_{n+2}(0)}{\bar\phi_{n+1}(0)}z\right)\Theta^*_{n+1}(z)\\ -\frac{\kappa_{n+1}}{\kappa_n}(z\Theta_n(z)-\Theta^*_n(z))-\frac{W(z)}{z}=0,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\Omega^*_n(z)-\Omega^*_{n+1}(z)+\frac{\kappa_{n+2}}{\kappa_{n+1}}\left(1+\frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\bar\phi_{n+2}(0)}{\kappa_{n+2}}z\right)\Theta^*_{n+1}(z)\\ +\frac{\phi_{n+1}(0)\bar\phi_{n+1}(0)}{\kappa_{n+1}\kappa_n}z\Theta_n(z)-\frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z)=0,\end{aligned} \tag{2.20}$$

$$\frac{\phi_{n+1}(0)}{\phi_n(0)}\Theta_n(z)-\frac{\kappa_n}{\kappa_{n-1}}z\Theta_{n-1}(z)=\frac{\bar\phi_{n+1}(0)}{\bar\phi_n(0)}z\Theta^*_n(z)-\frac{\kappa_n}{\kappa_{n-1}}\Theta^*_{n-1}(z), \tag{2.21}$$

$$\Omega^*_n(z)-\Omega_n(z)=-\frac{\kappa_{n+1}}{\kappa_n}(z\Theta_n(z)-\Theta^*_n(z))+n\frac{W(z)}{z}, \tag{2.22}$$

$$\Omega^*_n(z)+\Omega_n(z)=\frac{\kappa_n^2}{\kappa_{n+1}^2}\left[\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)}\Theta_{n+1}(z)+\frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z)\right]+\frac{W(z)}{z}. \tag{2.23}$$

In addition to the above coupled equations, evaluations of the coefficient functions at the singular points satisfy bilinear relations [12]

$$\Omega_n^2(z_j)=\frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z_j\Theta_n(z_j)\Theta_{n+1}(z_j)+V^2(z_j), \tag{2.24}$$

$$\Omega_n^{*2}(z_j)=\frac{\kappa_n\bar\phi_{n+2}(0)}{\kappa_{n+1}\bar\phi_{n+1}(0)}z_j\Theta^*_n(z_j)\Theta^*_{n+1}(z_j)+V^2(z_j), \tag{2.25}$$

(2.26)

$$\left[\Omega_{n-1}(z_j)-\frac{\kappa_{n-1}^2}{\kappa_n^2}\frac{\phi_{n+1}(0)}{\phi_n(0)}\Theta_n(z_j)\right]^2=\frac{\phi_{n+1}(0)\bar\phi_n(0)}{\kappa_n^2}\Theta_n(z_j)\Theta^*_{n-1}(z_j)+V^2(z_j),$$

$$\left[\Omega^*_{n-1}(z_j) - \frac{\kappa^2_{n-1}}{\kappa^2_n}\frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)} z_j \Theta^*_n(z_j)\right]^2 = \frac{\kappa_{n-1}\bar{\phi}_{n+1}(0)\phi_n(0)}{\kappa^3_n} z_j^2 \Theta^*_n(z_j)\Theta_{n-1}(z_j) + V^2(z_j), \tag{2.27}$$

$$\frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa^2_n} z_j \Theta_n(z_j)\Theta^*_n(z_j) + V^2(z_j) = \left[\Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j)\right]^2, \tag{2.28}$$

$$= \left[\Omega^*_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta^*_n(z_j)\right]^2. \tag{2.29}$$

It is these relations that lead directly to one of the pair of coupled discrete Painlevé equations. Deformation derivatives of the system (2.8) with respect to arbitrary trajectories of the singularities $z_j(t)$ are given in [12], leading to the Schlesinger equations from the theory of isomonodromic deformations of linear systems. The particular result that we require here is

$$\frac{1}{r_n}\frac{d}{dt} r_n = {}^1\!/\!_2 \sum_{j=1}^{m} \rho_j \frac{1}{z_j}\frac{d}{dt} z_j \frac{\Omega_{n-1}(z_j) - V(z_j)}{V(z_j)}, \tag{2.30}$$

$$\frac{1}{\bar{r}_n}\frac{d}{dt} \bar{r}_n = {}^1\!/\!_2 \sum_{j=1}^{m} \rho_j \frac{1}{z_j}\frac{d}{dt} z_j \frac{\Omega^*_{n-1}(z_j) + V(z_j)}{V(z_j)}. \tag{2.31}$$

3. The Semi-classical Class $m = 3$ and $\mathrm{dP_V}$

3.1. **Coupled Recurrences for the Reflection Coefficients.** Here we will consider the general theory from [12] revised above, applied to the simplest instance of the semi-classical weight (1.1), namely $m = 3$ singular points with two fixed at $z = 0, -1$ and the third variable at $z = -1/t$. Explicitly we consider the unitary group average (1.2) in the case of the weight (1.3). For this to make immediate sense we require $t \in \mathbb{T}$, but it can analytically continued off the unit circle. Note that when $t \in \mathbb{T}$, $\mu, \omega_1, \omega_2, \xi \in \mathbb{R}$ and $\xi < 1$ the weight (1.3) is real and positive. The corresponding Toeplitz determinant (2.4) is then hermitian and as a consequence $\bar{r}_n$ is the complex conjugate of r_n, but generally this is not the case. For parameters such that $\Re(\mu), \Re(\omega_1) > -1/2$, $t \in \mathbb{T}$ the Toeplitz matrix elements, w_{j-k} say, can be evaluated in terms of the Gauss hypergeometric function and so analytically continued into the general parameter space. There are several forms for this, which exhibit the analytic structure about the special points $t = 0, 1, \infty$. We will make note of two such expansions, relating to the special points $t = 0$ and $t = 1$.

Lemma 3.1. *For all points in parameter space that the functions below have meaning, the analytic continuation of the Toeplitz matrix elements w_n for the weight (1.3)*

is given by

(3.1)
$$
\begin{aligned}
t^{\mu} w_n = & \frac{\Gamma(2\omega_1+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n-\mu+\bar{\omega})} {}_2F_1(-2\mu, -n-\mu-\omega; 1-n-\mu+\bar{\omega}; t) \\
& + \frac{\xi}{2\pi i} e^{\pm\pi i(n+\mu-\bar{\omega})} \frac{\Gamma(2\mu+1)\Gamma(2\omega_1+1)}{\Gamma(2\mu+2\omega_1+2)} t^{n+\mu-\bar{\omega}} (1-t)^{2\mu+2\omega_1+1} \\
& \qquad \times {}_2F_1(2\mu+1, 1+n+\mu+\omega; 2\mu+2\omega_1+2; 1-t),
\end{aligned}
$$

where the $\pm$ *sign is taken accordingly as* $\mathrm{Im}(t) \gtrless 0$*. This can also be written as*

$$
\begin{aligned}
t^{\mu} w_n = & \left\{ 1 + \xi \frac{e^{\pm\pi i(n+\mu-\bar{\omega})}}{2i \sin\pi(n+\mu-\bar{\omega})} \right\} \frac{\Gamma(2\omega_1+1)}{\Gamma(1+n+\mu+\omega)\Gamma(1-n-\mu+\bar{\omega})} \\
& \times {}_2F_1(-2\mu, -n-\mu-\omega; 1-n-\mu+\bar{\omega}; t) \\
& - \xi \frac{e^{\pm\pi i(n+\mu-\bar{\omega})}}{2i \sin\pi(n+\mu-\bar{\omega})} \frac{\Gamma(2\mu+1)}{\Gamma(1+n+\mu-\bar{\omega})\Gamma(1-n+\mu+\bar{\omega})} \\
& \times t^{n+\mu-\bar{\omega}} (1-t)^{2\mu+2\omega_1+1} {}_2F_1(2\mu+1, 1+n+\mu+\omega; 1+n+\mu-\bar{\omega}; t).
\end{aligned}
\tag{3.2}
$$

Proof. This follows from the generalisation of the Euler integral for the Gauss hypergeometric function and consideration of the consistent phases for the branch cuts linking the singular points, see [19] pp. 91, section 17 "Verallgemeinerung der Eulersche Integrale". □

Remark 3.1. The first of these forms was given in [10].

For the description of (1.3) in terms of the characterisation (2.1) of general semi-classical weights we have

$$
m = 3, \quad \{z_j\}_{j=1}^{3} = \{0, -1, -1/t\}, \quad \{\rho_j\}_{j=1}^{3} = \{-\mu-\omega, 2\omega_1, 2\mu\}, \tag{3.3}
$$

$$
2V(0) = -(\mu+\omega)t^{-1}, \; W'(0) = t^{-1}, \quad V(-t^{-1}) = \mu\frac{1-t}{t^2}. \tag{3.4}
$$

Also, according to (2.13)-(2.16), the explicit forms of the coefficient functions in this case is

$$
\Theta_N(z) = \frac{\kappa_N}{\kappa_{N+1}} \left[(N+1+\mu+\bar{\omega})z - \frac{r_N}{r_{N+1}}(N+\mu+\omega)t^{-1} \right], \tag{3.5}
$$

$$
\Theta^*_N(z) = \frac{\kappa_N}{\kappa_{N+1}} \left[-\frac{\bar{r}_N}{\bar{r}_{N+1}}(N+\mu+\bar{\omega})z + (N+1+\mu+\omega)t^{-1} \right], \tag{3.6}
$$

$$
\begin{aligned}
\Omega_N(z) = & [1+\tfrac{1}{2}(\mu+\bar{\omega})]z^2 \\
& + \Big\{ (N+2+\mu+\bar{\omega})(1-r_{N+1}\bar{r}_{N+1})\frac{r_{N+2}}{r_{N+1}} - \frac{l_{N+1}}{\kappa_{N+1}} \\
& \qquad + [1+\tfrac{1}{2}(\mu+\bar{\omega})]\frac{1+t}{t} - \omega_1 - \frac{\mu}{t} \Big\} z - [N+\tfrac{1}{2}(\mu+\omega)]t^{-1},
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
(3.8)\quad & \Omega^*_N(z) = -\tfrac{1}{2}(\mu+\bar\omega)z^2 \\
& + \left\{ \frac{l_{N+1}}{\kappa_{N+1}} - (N+\mu+\bar\omega)(1-r_{N+1}\bar r_{N+1})\frac{\bar r_N}{\bar r_{N+1}} - \tfrac{1}{2}(\mu+\bar\omega)\frac{1+t}{t} + \omega_1 + \frac{\mu}{t} \right\} z \\
& \qquad\qquad + [N+1+\tfrac{1}{2}(\mu+\omega)]t^{-1}.
\end{aligned}
$$

We are seeking a closed system of recurrences for r_N and $\bar r_N$, since according to (2.6) these quantities determine the Toeplitz determinant, or equivalently the $U(N)$ average. One such recurrence is quite straight forward.

Lemma 3.2. *The reflection coefficients for the weight (1.3) satisfy the homogeneous second-order difference equation*

$$
\begin{aligned}
(3.9)\quad & (N+1+\mu+\bar\omega)tr_{N+1}\bar r_N - (N-1+\mu+\bar\omega)tr_N\bar r_{N-1} \\
& \qquad = (N+1+\mu+\omega)\bar r_{N+1}r_N - (N-1+\mu+\omega)\bar r_N r_{N-1}.
\end{aligned}
$$

Proof. This equation can be found immediately from the general theory of Section 2 in many ways. By equating coefficients of z in the functional-difference equation (2.19) using (3.5, 3.6,3.7,3.8), all are trivially satisfied except for the z coefficient, which is precisely (3.9). Similarly starting with (2.20) and employing (3.5, 3.6,3.8), one finds (3.9). Alternatively one could start with either (2.21) or (2.23) and arrive at the same result □

We will use this result in the derivation of a sequence of lemmas, which lead us to the sought closed system of coupled recurrences.

Corollary 3.1. *The sub-leading coefficients $l_N, \bar l_N$ satisfy the linear inhomogeneous equation*

$$
(3.10)\qquad (N+\mu+\bar\omega)tl_N - (N+\mu+\omega)\bar l_N = N\left[\mu(t-1)+\bar\omega-\omega t\right]\kappa_N.
$$

Proof. By substituting the general expression for the first difference of $l_N, \bar l_N$ using (2.3) in (3.9) one finds that it can be summed exactly to yield

$$
\begin{aligned}
(3.11)\quad & (N+1+\mu+\bar\omega)t\frac{l_{N+1}}{\kappa_{N+1}} - (N+1+\mu+\omega)\frac{\bar l_{N+1}}{\kappa_{N+1}} \\
& \qquad - (N+\mu+\bar\omega)t\frac{l_N}{\kappa_N} + (N+\mu+\omega)\frac{\bar l_N}{\kappa_N} = \mu(t-1)+\bar\omega-\omega t.
\end{aligned}
$$

This can be summed once more to yield the stated result. □

Remark 3.2. One could alternatively proceed via the Freud approach [14] (see also [13]) and consider the integral

$$
(3.12)\qquad \int_{\mathbb{T}} \frac{dz}{2\pi i z}(1+z)(1+tz)\left[-\frac{\mu+\omega}{z} + \frac{2\omega_1}{1+z} + \frac{2\mu t}{1+tz}\right] w(z)\phi_N(z)\bar\phi_N(\bar z).
$$

Here we recognise the logarithmic derivative of the weight function in the integrand

$$\frac{w'}{w} = -\frac{\mu+\omega}{z} + \frac{2\omega_1}{1+z} + \frac{2\mu t}{1+tz}, \tag{3.13}$$

and by evaluating the integral in the two ways we find a linear equation for l_N, namely (3.11).

Lemma 3.3. *The sub-leading coefficients are related to the reflection coefficients by*

$$\begin{aligned} &\bar{l}_N/\kappa_N + tl_N/\kappa_N - N(t+1) \\ &= \frac{1-r_N\bar{r}_N}{r_N}\left[(N+1+\mu+\bar{\omega})tr_{N+1} + (N-1+\mu+\omega)r_{N-1}\right], &(3.14)\\ &= \frac{1-r_N\bar{r}_N}{\bar{r}_N}\left[(N+1+\mu+\omega)\bar{r}_{N+1} + (N-1+\mu+\bar{\omega})t\bar{r}_{N-1}\right]. &(3.15)\end{aligned}$$

Proof. The first relation follows from a comparison of the coefficients of z for $\Omega_N(z)$ given the two distinct expansions, the first by (2.14) which reduces to (3.7) and the second by the specialisation of (2.16). The second relation follows from identical arguments applied to $\Omega^*_N(z)$ or by employing (3.9) in the first relation. □

Remark 3.3. The first result appears in the Magnus derivation [20] for the generalised Jacobi weight, with $\theta_1 = \pi - \phi, \theta_2 = \pi$, $\alpha = \mu, \beta = \omega_1, \gamma = -\omega_2$. Then Equation (14) of that work is precisely (3.14).

Remark 3.4. The Magnus relation (3.14) can also be found by employing the Freud method. In this one uses integration by parts on the integral

$$\int_{\mathbb{T}} \frac{dz}{2\pi iz} z^{-1}(1+z)(1+tz)w'(z)\phi_{N+1}(z)\bar{\phi}_N(\bar{z}), \tag{3.16}$$

and in the term involving $\phi'_{N+1}(z)$ one employs (2.12) for the derivative and (3.5), (3.7) for the coefficient functions. Equating this expression to a direct evaluation of the integral then yields (3.14).

Lemma 3.4. *The sub-leading coefficient l_N can be expressed in terms of the reflection coefficients in the following ways*

$$\begin{aligned} 2t\frac{l_N}{\kappa_N} =&(N+1+\mu+\bar{\omega})t(\frac{r_{N+1}}{r_N} - r_{N+1}\bar{r}_N) + (N-1+\mu+\omega)\frac{r_{N-1}}{r_N} &(3.17)\\ &-(N-1+\mu+\bar{\omega})r_N\bar{r}_{N-1} + (N+\mu-\omega)t + N - \mu + \bar{\omega}, \\ =&(N+1+\mu+\omega)\frac{\bar{r}_{N+1}}{\bar{r}_N} + (N-1+\mu+\bar{\omega})t(\frac{\bar{r}_{N-1}}{\bar{r}_N} - r_N\bar{r}_{N-1}) &(3.18)\\ &-(N+1+\mu+\bar{\omega})tr_{N+1}\bar{r}_N + (N+\mu-\omega)t + N - \mu + \bar{\omega}, \end{aligned}$$

as well as analogous expressions for $\bar{l}_N$.

Proof. The first expression follows from a comparison of the z^0 coefficients for $\Theta_N(z)$ evaluated using both (2.13) and (2.15). The second relation follows from an applying the same reasoning to $\Theta^*_N(z)$. □

We will refer to the order of a system of coupled difference equations with two variables, $r_n, \bar{r}_n$ say, as q/p where $q \in \mathbb{Z}_{\geq 0}$ refers to the order of r_n and $p \in \mathbb{Z}_{\geq 0}$ refers to the order of $\bar{r}_n$.

Corollary 3.2. *The reflection coefficients of the OPS for the weight (1.3) satisfy the* $2/2$ *order recurrence relations*

$$
\begin{aligned}
&t r_N \bar{r}_{N-1} + r_{N-1}\bar{r}_N - t - 1 \\
&= \frac{1 - r_N\bar{r}_N}{r_N}\left[(N+1+\mu+\bar{\omega})t r_{N+1} + (N-1+\mu+\omega) r_{N-1}\right] \\
&\quad -\frac{1 - r_{N-1}\bar{r}_{N-1}}{\bar{r}_{N-1}}\left[(N+\mu+\omega)\bar{r}_N + (N-2+\mu+\bar{\omega})t\bar{r}_{N-2}\right], \qquad (3.19)\\
&= \frac{1 - r_N\bar{r}_N}{\bar{r}_N}\left[(N+1+\mu+\omega)\bar{r}_{N+1} + (N-1+\mu+\bar{\omega})t\bar{r}_{N-1}\right] \\
&\quad -\frac{1 - r_{N-1}\bar{r}_{N-1}}{r_{N-1}}\left[(N+\mu+\bar{\omega})t r_N + (N-2+\mu+\omega) r_{N-2}\right], \qquad (3.20)
\end{aligned}
$$

and those specifying the solution for (2.6) have the initial values

$$r_0 = \bar{r}_0 = 1, \quad r_1 = -w_{-1}/w_0, \quad \bar{r}_1 = -w_1/w_0, \qquad (3.21)$$

where the Toeplitz matrix elements are given in (3.2).

Proof. Solving (3.14) for the combination of $l_N, \bar{l}_N$ and differencing this, one arrives at (3.19). This however is of order $3/1$ but by employing (3.9) we can reduce the order in r_N of the recurrence to second order. The other member of the pair (3.20) is found in an identical manner starting with (3.15). □

Remark 3.5. The second-order difference (3.19) also follows immediately from equating the polynomials in z arising in the functional-difference (2.17), after employing (3.5,3.7). The other member of the pair, (3.20), follows from the functional-difference (2.18), after using (3.6,3.8).

3.2. **Relationship to recurrences of Adler and van Moerbeke.** In their most general example, Adler and van Moerbeke [1] also considered recurrences from the $U(N)$ average with weight (1.3) ($\xi = 0$ case). Their study proceeded via the viewpoint of the Toeplitz lattice and Virasoro constraints. In terms of their variables we should set $P_1 = P_2 = 0, d_1 = t^{-1/2}, d_2 = t^{1/2}$, and without loss of generality $\gamma''_1 = \gamma'_2 = 0$. For the other parameters $\gamma = \mu - \omega, \gamma'_1 = 2\omega_1, \gamma''_2 = 2\mu$. There is a slight difference in the dependent variables due to the additional factor of t, so that we have the identification $x_N = (-1)^N t^{N/2} r_N$, $y_N = (-1)^N t^{-N/2}\bar{r}_N$ and

$v_N = 1 - r_N\bar{r}_N$. Generalising their working one finds that their Equation (0.0.14) implies

$$(3.22)\quad \begin{aligned} &-(N+1+\mu+\bar{\omega})x_{N+1}y_N + (N+1+\mu+\omega)x_N y_{N+1} \\ &\qquad + (N-1+\mu+\bar{\omega})x_N y_{N-1} - (N-1+\mu+\omega)x_{N-1}y_N = 0. \end{aligned}$$

Now by transforming to our $r_N, \bar{r}_N$ and employing (2.3) one finds this is precisely (3.9), which we showed is solved by (3.10). Their inhomogeneous Equation (0.0.15) now takes the form

$$(3.23)\quad \begin{aligned} &-v_N\left[(N+1+\mu+\bar{\omega})x_{N+1}y_{N-1} + N+\mu+\omega\right] \\ &\quad + v_{N-1}\left[(N-2+\mu+\bar{\omega})x_N y_{N-2} + N-1+\mu+\omega\right] \\ &\qquad + x_N y_{N-1}(x_N y_{N-1} + t^{1/2} + t^{-1/2}) \\ &\qquad = -v_1\left[(2+\mu+\bar{\omega})x_2 + 1+\mu+\omega\right] + x_1(x_1 + t^{1/2} + t^{-1/2}). \end{aligned}$$

Upon recasting this into our variables and manipulating, it then becomes

$$\begin{aligned} &t r_N\bar{r}_{N-1} + \bar{r}_N r_{N-1} - t - 1 \\ &\quad - \frac{1-r_N\bar{r}_N}{r_N}\left[(N+1+\mu+\bar{\omega})t r_{N+1} + (N-1+\mu+\omega)r_{N-1}\right] \\ &\quad + \frac{1-r_{N-1}\bar{r}_{N-1}}{\bar{r}_{N-1}}\left[(N+\mu+\omega)\bar{r}_N + (N-2+\mu+\bar{\omega})t\bar{r}_{N-2}\right] \\ &\qquad = \frac{1-(1-r_1\bar{r}_1)\left[(2+\mu+\bar{\omega})t r_2 + 1+\mu+\omega\right] + r_1(t r_1 - t - 1)}{r_N\bar{r}_{N-1}}. \end{aligned}$$

However using the identity

$$\begin{aligned} &c\,{}_2F_1(a,b;c;x) \\ &= [c+(1+b-a)x]\,{}_2F_1(a,b+1;c+1;x) - \frac{b+1}{c+1}(1+c-a)x\,{}_2F_1(a,b+2;c+2;x), \end{aligned}$$

we note that the right-hand side is identically zero for the initial conditions (3.21) and the recurrence is not genuinely inhomogeneous, thus yielding our first relation above, (3.19).

3.3. Transformation of the recurrences to $\mathrm{dP_V}$.

We seek recurrences for $r_N, \bar{r}_N$ which are of the form of the discrete Painlevé system (1.6), (1.7). For this purpose a number of distinct forms of the former will be presented.

Proposition 3.1. *The reflection coefficients satisfy a system of a $2/0$ order recurrence relation*

$$
\begin{aligned}
(3.24)\quad & \Big\{(1-r_N\bar{r}_N)\left[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\right.\\
&\left. -(N+\mu+\omega)(N-1+\mu+\omega)r_{N-1}\right]+N(N+2\omega_1)(t-1)r_N\Big\}\\
&\times\Big\{(1-r_N\bar{r}_N)\left[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\right.\\
&\left.-(N+\mu+\omega)(N-1+\mu+\omega)r_{N-1}\right]+(N+2\mu)(N+2\mu+2\omega_1)(t-1)r_N\Big\}\\
&=-(2N+2\mu+2\omega_1)^2t(1-r_N\bar{r}_N)\\
&\times\left[(N+1+\mu+\bar{\omega})r_{N+1}+(N+\mu+\omega)r_N\right]\\
&\times\left[(N+\mu+\bar{\omega})r_N+(N-1+\mu+\omega)r_{N-1}\right],
\end{aligned}
$$

and a $0/2$ order recurrence relation which is just (3.24) with the replacements $\omega\leftrightarrow\bar{\omega}$ and $t^{\pm1/2}r_j\mapsto t^{\mp1/2}\bar{r}_j$

Proof. Consider first the specialisation of (2.24) to our weight at hand at the singular point $z=-1$, and we have

$$
\begin{aligned}
(3.25)\quad & \left\{\frac{l_N}{\kappa_N}-Nt^{-1}-(N+1+\mu+\bar{\omega})\frac{\kappa_{N-1}^2}{\kappa_N^2}\frac{r_{N+1}}{r_N}+\omega_1(1-t^{-1})\right\}^2\\
&+\frac{\kappa_{N-1}^2}{\kappa_N^2}\left[N+\mu+\bar{\omega}+\frac{(N-1+\mu+\omega)}{t}\frac{r_{N-1}}{r_N}\right]\\
&\times\left[\frac{(N+\mu+\omega)}{t}+(N+1+\mu+\bar{\omega})\frac{r_{N+1}}{r_N}\right]=\omega_1^2\left(\frac{t-1}{t}\right)^2,
\end{aligned}
$$

by using (3.5,3.7). Similarly (2.24) evaluated at $z=-1/t$ yields

$$
\begin{aligned}
(3.26)\quad & \left\{\frac{l_N}{\kappa_N}-N-(N+1+\mu+\bar{\omega})\frac{\kappa_{N-1}^2}{\kappa_N^2}\frac{r_{N+1}}{r_N}+\mu(t^{-1}-1)\right\}^2\\
&+\frac{\kappa_{N-1}^2}{\kappa_N^2}t^{-1}\left[N+\mu+\bar{\omega}+(N-1+\mu+\omega)\frac{r_{N-1}}{r_N}\right]\\
&\times\left[N+\mu+\omega+(N+1+\mu+\bar{\omega})\frac{r_{N+1}}{r_N}\right]=\mu^2\left(\frac{t-1}{t}\right)^2.
\end{aligned}
$$

The first relation follow by eliminating l_N between (3.25) and (3.26), whereas the second follows from an identical analysis to that employed in the proof of Proposition 3.4 but starting with the bilinear identity (2.25). □

Proposition 3.2. *The reflection coefficients also satisfy a system of 1/1 order recurrence relations the first of which is*

$$
\begin{aligned}
&\Big\{ -(N+\mu+\omega)(1-r_N\bar{r}_N)t\left[(N+1+\mu+\bar{\omega})r_{N+1}\bar{r}_N+(N-1+\mu+\bar{\omega})r_N\bar{r}_{N-1}\right] \\
&\quad +2(N+\mu+\omega)^2r_N^2\bar{r}_N^2-(N+\mu+\omega)^2(t+1)r_N\bar{r}_N-2(N+\mu+\omega)\bar{\omega}(t-1)r_N\bar{r}_N \\
&\qquad +(\mu-\bar{\omega})(\mu+\bar{\omega})(t-1)\Big\} \\
&\times \\
&\Big\{ -(N+\mu+\omega)(1-r_N\bar{r}_N)t\left[(N+1+\mu+\bar{\omega})r_{N+1}\bar{r}_N+(N-1+\mu+\bar{\omega})r_N\bar{r}_{N-1}\right] \\
&\quad +2(N+\mu+\omega)^2r_N^2\bar{r}_N^2-(N+\mu+\omega)^2(t+1)r_N\bar{r}_N+2(N+\mu+\omega)\omega(t-1)r_N\bar{r}_N \\
&\qquad +(\mu-\omega)(\mu+\omega)(t-1)\Big\} \\
&= -\left[2(N+\mu+\omega)r_N\bar{r}_N+\bar{\omega}-\omega\right]^2(1-r_N\bar{r}_N) \\
&\quad \times\left[(N+1+\mu+\bar{\omega})tr_{N+1}+(N+\mu+\omega)r_N\right] \\
&\qquad \times\left[(N+\mu+\omega)\bar{r}_N+(N-1+\mu+\bar{\omega})t\bar{r}_{N-1}\right],
\end{aligned}
\tag{3.27}
$$

and the second is obtained from (3.27) with the replacements $\omega \leftrightarrow \bar{\omega}$ *and* $t^{\pm 1/2}r_j \leftrightarrow t^{\mp 1/2}\bar{r}_j$.

Proof. The specialisation of (2.26) to the weight (1.3) evaluated at the singular point $z=-1$ is

$$
\begin{aligned}
&\left\{\frac{l_N}{\kappa_N}-Nt^{-1}+(N+\mu+\omega)t^{-1}\frac{\kappa_{N-1}^2}{\kappa_N^2}+\omega_1(1-t^{-1})\right\}^2 \\
&\quad +\frac{\kappa_{N-1}^2}{\kappa_N^2}\left[(N+1+\mu+\bar{\omega})r_{N+1}+(N+\mu+\omega)t^{-1}r_N\right] \\
&\quad \times\left[(N-1+\mu+\bar{\omega})\bar{r}_{N-1}+(N+\mu+\omega)t^{-1}\bar{r}_N\right]=\omega_1^2\left(\frac{t-1}{t}\right)^2,
\end{aligned}
\tag{3.28}
$$

by using (3.5,3.7). Similarly (2.26) evaluated at $z=-1/t$ yields

$$
\begin{aligned}
&\left\{\frac{l_N}{\kappa_N}-N+(N+\mu+\omega)\frac{\kappa_{N-1}^2}{\kappa_N^2}+\mu(t^{-1}-1)\right\}^2 \\
&\quad +\frac{\kappa_{N-1}^2}{\kappa_N^2}\left[(N+1+\mu+\bar{\omega})r_{N+1}+(N+\mu+\omega)r_N\right] \\
&\quad \times\left[(N-1+\mu+\bar{\omega})\bar{r}_{N-1}+(N+\mu+\omega)\bar{r}_N\right]=\mu^2\left(\frac{t-1}{t}\right)^2.
\end{aligned}
\tag{3.29}
$$

Again eliminating l_N between these two equations yields the recurrence relation (3.27). The second follows in the same way starting with (2.27). □

Proposition 3.3. *The reflection coefficients satisfy an alternative system of* $1/1$ *order recurrence relations the first of which is*

$$\begin{aligned}(3.30)\quad &\Big[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\bar{r}_N\\ &-(N+1+\mu+\omega)(N+\mu+\omega)\bar{r}_{N+1}r_N+(\bar{\omega}-\mu)(\bar{\omega}+\mu)(t-1)\Big]\\ &\times\Big[(N+1+\mu+\bar{\omega})(N+\mu+\bar{\omega})tr_{N+1}\bar{r}_N\\ &-(N+1+\mu+\omega)(N+\mu+\omega)\bar{r}_{N+1}r_N+(\omega-\mu)(\omega+\mu)(t-1)\Big]\\ &=(\bar{\omega}-\omega)^2\left[(N+1+\mu+\bar{\omega})tr_{N+1}+(N+\mu+\omega)r_N\right]\\ &\times\left[(N+1+\mu+\omega)\bar{r}_{N+1}+(N+\mu+\bar{\omega})t\bar{r}_N\right],\end{aligned}$$

and the second is again obtained from (3.30) with the replacements $\omega\leftrightarrow\bar{\omega}$ *and* $t^{\pm 1/2}r_j\leftrightarrow t^{\mp 1/2}\bar{r}_j$.

Proof. The specialisation of (2.28) to the weight (1.3) evaluated at the singular point $z=-1$ is

$$\begin{aligned}(3.31)\quad &\left\{\frac{\bar{l}_{N+1}}{\kappa_{N+1}}+(N+\mu+\omega)\bar{r}_{N+1}r_N+\omega_1+(\mu-i\omega_2)t\right\}^2\\ &=\left[(N+1+\mu+\bar{\omega})tr_{N+1}+(N+\mu+\omega)r_N\right]\\ &\times\left[(N+1+\mu+\omega)\bar{r}_{N+1}+(N+\mu+\bar{\omega})t\bar{r}_N\right]+\omega_1^2(t-1)^2,\end{aligned}$$

by using (3.5,3.7). Similarly (2.28) evaluated at $z=-1/t$ yields

$$\begin{aligned}(3.32)\quad &\left\{\frac{\bar{l}_{N+1}}{\kappa_{N+1}}+(N+\mu+\omega)\bar{r}_{N+1}r_N+\bar{\omega}+\mu t\right\}^2\\ &=t\left[(N+1+\mu+\bar{\omega})r_{N+1}+(N+\mu+\omega)r_N\right]\\ &\times\left[(N+1+\mu+\omega)\bar{r}_{N+1}+(N+\mu+\bar{\omega})\bar{r}_N\right]+\mu^2(t-1)^2,\end{aligned}$$

Again eliminating $\bar{l}_{N+1}$ between these two equations yields the recurrence relation (3.30). The second follows in the same way starting with (2.29). □

Remark 3.6. Note that the recurrence system (3.24) and its partner is quadratic in r_{N+1}, r_{N-1} and $\bar{r}_{N+1}$, $\bar{r}_{N-1}$, the system (3.27) and its partner is also quadratic in $r_{N+1},\bar{r}_{N-1}$ and $\bar{r}_{N+1}$, r_{N-1}, and likewise (3.30) is quadratic in r_{N+1}, $\bar{r}_{N+1}$. This renders them less useful in practical iterations than the higher order systems that are linear in the highest difference. By raising the order of one of the variables by one we can obtain a recurrence linear in the highest difference.

Corollary 3.3. *The reflection coefficients satisfy a system of a* $2/1$ *order recurrence relation*

$$
\begin{aligned}
(3.33)\quad & (N+1+\mu+\bar{\omega})(\bar{\omega}-\omega)t(1-r_N\bar{r}_N)r_{N+1} \\
& + (N-1+\mu+\omega)[2(N+\mu+\omega)r_N\bar{r}_N+\bar{\omega}-\omega]r_{N-1} \\
& - (N-1+\mu+\bar{\omega})(2N+2\mu+2\omega_1)tr_N^2\bar{r}_{N-1} \\
& + \big[(\bar{\omega}-\omega)N(t+1)-(2\mu+2\omega_1)[\mu(1-t)+\omega t-\bar{\omega}]\big]r_N = 0,
\end{aligned}
$$

and a $1/2$ *order recurrence relation which is again obtained from (3.33) with the replacements* $\omega \leftrightarrow \bar{\omega}$ *and* $t^{\pm 1/2}r_j \leftrightarrow t^{\mp 1/2}\bar{r}_j$.

Proof. The solutions for the sub-leading coefficient $l_N, \bar{l}_N$ that arise from the simultaneous solution of (3.28,3.29) and (3.31,3.32) respectively are given by

$$
\begin{aligned}
(3.34)\quad t\frac{l_N}{\kappa_N} = \Big\{ & (N+\mu+\omega)t(1-r_N\bar{r}_N) \\
& \times [(N+1+\mu+\bar{\omega})r_{N+1}\bar{r}_N+(N-1+\mu+\bar{\omega})r_N\bar{r}_{N-1}] \\
& + (N+\mu+\omega)[N(t+1)-\mu(1-t)-\omega t+\bar{\omega}]r_N\bar{r}_N+(\omega+\mu)[\mu(1-t)+\omega t-\bar{\omega}]\Big\} \\
& \div [2(N+\mu+\omega)r_N\bar{r}_N+\bar{\omega}-\omega],
\end{aligned}
$$

$$
\begin{aligned}
(3.35)\quad & \\
t\frac{l_N}{\kappa_N} = \Big\{ & (N+\mu+\omega)\,[(N-1+\mu+\bar{\omega})tr_N\bar{r}_{N-1}-(N-1+\mu+\omega)\bar{r}_Nr_{N-1}] \\
& + (\omega+\mu)[\mu(1-t)+\omega t-\bar{\omega}]\Big\}\Big/(\bar{\omega}-\omega), \quad \text{if } \bar{\omega}\neq\omega,
\end{aligned}
$$

and the corresponding expression for $\bar{l}_N/\kappa_N$ under the above replacements. Equating these two forms then leads to (3.33). □

The systems of recurrences that we have found are in fact equivalent to the discrete Painlevé equation associated with the degeneration of the rational surface $D_4^{(1)} \to D_5^{(1)}$ and we give our first demonstration of this fact here.

Proposition 3.4. *The* N*-recurrence for the reflection coefficients of the orthogonal polynomial system with the weight (1.3) is governed by either of two systems of coupled first order discrete Painlevé equations (1.6), (1.7). This first is*

$$
(3.36)\qquad g_{N+1}g_N = t\frac{(f_N+N)(f_N+N+2\mu)}{f_N(f_N-2\omega_1)},
$$

$$
(3.37)\qquad f_N+f_{N-1} = 2\omega_1+\frac{N-1+\mu+\omega}{g_N-1}+\frac{(N+\mu+\bar{\omega})t}{g_N-t},
$$

subject to the initial conditions

$$
(3.38)\qquad g_1 = t\frac{\mu+\omega+(1+\mu+\bar{\omega})r_1}{\mu+\omega+(1+\mu+\bar{\omega})tr_1}, \quad f_0 = 0.
$$

The transformations relating these variables to the reflection coefficients are given by

$$g_N = t\frac{N-1+\mu+\omega+(N+\mu+\bar{\omega})\dfrac{r_N}{r_{N-1}}}{N-1+\mu+\omega+(N+\mu+\bar{\omega})t\dfrac{r_N}{r_{N-1}}}, \tag{3.39}$$

$$f_N = \frac{1}{1-t}\left[t\frac{l_N}{\kappa_N} - N - (N+1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{r_{N+1}}{r_N}\right]. \tag{3.40}$$

The second system is

$$\bar{g}_{N+1}\bar{g}_N = t^{-1}\frac{(\bar{f}_N+N)(\bar{f}_N+N+2\omega_1)}{\bar{f}_N(\bar{f}_N-2\mu)}, \tag{3.41}$$

$$\bar{f}_N + \bar{f}_{N-1} = 2\mu + \frac{N+\mu+\omega}{\bar{g}_N-1} + \frac{(N-1+\mu+\bar{\omega})t^{-1}}{\bar{g}_N-t^{-1}}, \tag{3.42}$$

subject to the initial conditions

$$\bar{g}_1 = \frac{\mu+\bar{\omega}+(1+\mu+\omega)t^{-1}\bar{r}_1}{\mu+\bar{\omega}+(1+\mu+\omega)\bar{r}_1}, \quad \bar{f}_0 = 0. \tag{3.43}$$

The transformations relating these variables to the reflection coefficients are given by

$$\bar{g}_N = \frac{N-1+\mu+\bar{\omega}+(N+\mu+\omega)t^{-1}\dfrac{\bar{r}_N}{\bar{r}_{N-1}}}{N-1+\mu+\bar{\omega}+(N+\mu+\omega)\dfrac{\bar{r}_N}{\bar{r}_{N-1}}}, \tag{3.44}$$

$$\bar{f}_N = \frac{1}{1-t}\left[-t\frac{l_N}{\kappa_N} + Nt + (N-1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{\bar{r}_{N-1}}{\bar{r}_N}\right]. \tag{3.45}$$

Proof. Consolidating each of (3.25) and (3.26) into two terms and taking their ratio then leads to (3.36) after utilising the definitions (3.39,3.40). The second member of the recurrence system (3.37) follows from the relation

$$\begin{aligned} &\frac{l_{N+1}}{\kappa_{N+1}} + \frac{l_N}{\kappa_N} \\ &= (N+2+\mu+\bar{\omega})(1-r_{N+1}\bar{r}_{N+1})\frac{r_{N+2}}{r_{N+1}} + (N+\mu+\omega)t^{-1}\frac{r_N}{r_{N+1}} - (N+1+\mu+\bar{\omega})r_{N+1}\bar{r}_N \\ &\qquad - 2\omega_1 - 2\mu t^{-1} + (N+1+\mu+\bar{\omega})(1+t^{-1}), \end{aligned} \tag{3.46}$$

which results from a combination of (3.17) and (2.3), and the definition (3.40). All the results for the second system follow by applying identical reasoning starting with (2.25). □

3.4. **Evaluations in terms of generalised hypergeometric functions.** In the special case $\xi = 0$ of the $U(N)$ average (1.2) with weight (1.3), we know from [11] that an evaluation in terms of a generalised hypergeometric function ${}_2F_1^{(1)}$ is possible. Let us first recall the definition of the latter. Given a partition $\kappa =$

$(\kappa_1, \kappa_2, \ldots, \kappa_N)$ such that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N \geq 0$ one defines the generalised, multi-variable hypergeometric function through a series representation [24, 16]

(3.47)
$$ {}_pF_q^{(1)}(a_1, \ldots, a_p; b_1, \ldots, b_q; t_1, \ldots, t_N) = \sum_{\kappa \geq 0}^{\infty} \frac{[a_1]_\kappa^{(1)} \cdots [a_p]_\kappa^{(1)}}{[b_1]_\kappa^{(1)} \cdots [b_q]_\kappa^{(1)}} \frac{s_\kappa(t_1, \ldots, t_N)}{h_\kappa}, $$

for $p, q \in \mathbb{Z}_{\geq 0}$. Here the generalised Pochhammer symbols are

$$ [a]_\kappa^{(1)} := \prod_{j=1}^{N} (a - j + 1)_{\kappa_j}, \tag{3.48} $$

the hook length is

$$ h_\kappa = \prod_{(i,j) \in \kappa} [a(i,j) + l(i,j) + 1], \tag{3.49} $$

where $a(i,j), l(i,j)$ are the arm and leg lengths of the (i,j)th box in the Young diagram of the partition κ, and $s_\kappa(t_1, \ldots, t_N)$ is the Schur symmetric polynomial of N variables. The superscript (1) distinguishes these functions from the single variable $N = 1$ functions and also indicates that they are a special case of a more general function parameterised by an arbitrary complex number $d \neq 1$.

With this definition recalled, the result of [11] (see also [8]) reads

$$ \begin{aligned} \Big\langle \prod_{l=1}^{N} z_l^{-\mu-\omega} (1 + z_l)^{2\omega_1} (1 + t z_l)^{2\mu} \Big\rangle_{U(N)} &= \prod_{j=0}^{N-1} \frac{j! \Gamma(2\omega_1 + j + 1)}{\Gamma(1 + \mu + \omega + j) \Gamma(1 - \mu + \bar{\omega} + j)} \\ &\quad \times {}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \ldots, t_N)|_{t_1 = \ldots = t_N = t}, \end{aligned} \tag{3.50} $$

subject to $\Re(\omega_1) > -1/2$ and $|t| < 1$. The new observation we make here is that the reflection coefficients determining (3.50) can similarly be written in terms of the ${}_2F_1^{(1)}$ function.

Proposition 3.5. *With $I_N[w]$ in (2.6) given by (3.50), the corresponding reflection coefficients are given by*

$$ \begin{aligned} r_N = (-1)^N &\frac{(\mu + \omega)_N}{(1 - \mu + \bar{\omega})_N} \\ &\times \left. \frac{{}_2F_1^{(1)}(-2\mu, 1 - \mu - \omega; N + 1 - \mu + \bar{\omega}; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \ldots, t_N)} \right|_{t_1 = \ldots = t_N = t}, \end{aligned} \tag{3.51} $$

$$ \begin{aligned} \bar{r}_N = (-1)^N &\frac{(-\mu + \bar{\omega})_N}{(1 + \mu + \omega)_N} \\ &\times \left. \frac{{}_2F_1^{(1)}(-2\mu, -1 - \mu - \omega; N - 1 - \mu + \bar{\omega}; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(-2\mu, -\mu - \omega; N - \mu + \bar{\omega}; t_1, \ldots, t_N)} \right|_{t_1 = \ldots = t_N = t}. \end{aligned} \tag{3.52} $$

Proof. For $\epsilon = 0, \pm 1$ we define the Toeplitz determinants or $U(N)$ averages

$$I_n^\epsilon[w] := \det\left[\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^{\epsilon-j+k}\right]_{0\le j,k\le n-1} = \Big\langle \prod_{l=1}^n z_l^\epsilon w(z_l)\Big\rangle_{U(n)}. \tag{3.53}$$

From the Szegö theory we know

$$r_n = (-1)^n \frac{I_n^1[w]}{I_n^0[w]}, \quad \bar{r}_n = (-1)^n \frac{I_n^{-1}[w]}{I_n^0[w]}. \tag{3.54}$$

The result now follows from (3.50). □

According to the definition (3.47), ${}_2F_1^{(1)}$ is normalised so that

$${}_2F_1^{(1)}(a, b; c; t_1, \dots, t_N)|_{t_1=\dots=t_N=0} = 1. \tag{3.55}$$

At the special point when each t_i equals unity, the analog of the Gauss summation gives the gamma function evaluation [24]

$$\begin{aligned} {}_2F_1^{(1)}&(-2\mu, -\mu-\omega; N-\mu+\bar{\omega}; t_1, \dots, t_N)|_{t_1=\dots=t_N=1} \\ &= \prod_{j=1}^N \frac{\Gamma(j+2\mu+2\omega_1)\Gamma(j-\mu+\bar{\omega})}{\Gamma(j+2\omega_1)\Gamma(j+\mu+\bar{\omega})}, \end{aligned} \tag{3.56}$$

when $\Re(\mu+\omega_1) > -1/2$, $\Re(-\mu+\bar{\omega}) > -1$.

The result (3.55) tells us that at $t = 0$

$$r_N = (-1)^N \frac{(\mu+\omega)_N}{(1-\mu+\bar{\omega})_N}, \quad \bar{r}_N = (-1)^N \frac{(-\mu+\bar{\omega})_N}{(1+\mu+\omega)_N}, \tag{3.57}$$

while (3.56) tells us that at $t = 1$

$$r_N = (-1)^N \frac{(\mu+\omega)_N}{(1+\mu+\bar{\omega})_N}, \quad \bar{r}_N = (-1)^N \frac{(\mu+\bar{\omega})_N}{(1+\mu+\omega)_N}. \tag{3.58}$$

We note that the corresponding values of l_N can be calculated as

$$\frac{l_N}{\kappa_N} = -\frac{(\mu+\omega)N}{(N-\mu+\bar{\omega})}, \quad \frac{l_N}{\kappa_N} = -\frac{(\mu+\omega)N}{(N+\mu+\bar{\omega})}. \tag{3.59}$$

3.5. **Two remarks.** We conclude this section with two remarks. The first relates to the limit transition from the average (1.2) with weight (1.3) to the average

$$\Big\langle \prod_{l=1}^N z_l^{(\mu-\nu)/2} |1+z_l|^{\mu+\nu} e^{tz_l} \Big\rangle_{U(N)}, \tag{3.60}$$

studied from the viewpoint of bi-orthogonal polynomials in [13]. The latter can be obtained as a degeneration of the former by the replacements $\omega+\mu \mapsto \nu, \bar{\omega}-\mu \mapsto \mu, t \mapsto t/2\mu$ and then taking the limit $\mu \to \infty$. The coefficients of the orthogonal polynomials r_N, l_N remain of O(1) in this limit. Then we see the explicit degeneration of the following equations - (3.10) $\to$ Equation (4.23)[13], the recurrence relations (3.27) $\to$ Equation (4.60)[13], (3.33) $\to$ Equation (4.9)[13] and its conjugate

to Equation (4.10)[13] modulo the identity Equation (4.5)[13], and the hypergeometric functions (3.50) $\to$ Equation (4.24)[13], (3.51) $\to$ Equation (4.26)[13], and (3.52) $\to$ Equation (4.27)[13].

The second remark relates to the choices of parameters, noted below (1.3), for which the weight is real and positive, and consequently $\bar{r}_n = r_n$. Let us suppose furthermore that $\omega_2 = 0$, $\xi = 0$. Then with $t = e^{i\phi}$, (1.3) reads

$$w(e^{i\theta}) = |2\cos\tfrac{1}{2}\theta|^{2\omega}|2\cos\tfrac{1}{2}(\theta+\phi)|^{2\mu}. \tag{3.61}$$

It follows from this that $w_{-1} = tw_1$, and this in (3.21) tells us that $\bar{r}_1 = tr_1$. This initial value, together with the initial value given by the first equation in (3.21), allows a structural formula for r_n in this case to be obtained.

Corollary 3.4. *Let (1.3) be specialised to (3.61). Then the reflection coefficient r_n, which is related to $\bar{r}_n$ by complex conjugation, has the form $r_n = t^{-n/2}x_n$, where the x_n are real.*

Proof. Setting $\omega_2 = 0$ in (3.9) we note this can be rearranged as

$$(n+1+\mu+\omega)\left[t\frac{r_{n+1}}{r_n} - \frac{\bar{r}_{n+1}}{\bar{r}_n}\right] + (n-1+\mu+\omega)\left[\frac{r_{n-1}}{r_n} - t\frac{\bar{r}_{n-1}}{\bar{r}_n}\right] = 0. \tag{3.62}$$

It is easy to verify that this has the solution $\bar{r}_n = t^n r_n$ which is furthermore consistent with the initial conditions. The result now follows from the fact that $\bar{r}_n$ is the complex conjugate of r_n. $\square$

4. The τ-function Theory for $\mathrm{P_{VI}}$

In a previous study [10] the average (1.2) with weight (1.3) has been shown to satisfy recurrences involving $\mathrm{dP_V}$, distinct from those isolated in Proposition 3.4. Here we will present the explicit transformation between the two systems. We will also show how the relationship between the $U(N)$ average and an average over the Jacobi unitary ensemble, studied from the viewpoint of τ-function theory in [11], can be used to deduce a further characterisation involving $\mathrm{dP_V}$.

The study [10] is based on the Okamoto τ-function theory of Painlevé systems. In the case of $\mathrm{P_{VI}}$, a Hamiltonian H is defined in terms of coordinate and momenta variables q, p, a time variable t, and parameters $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$ subject to the constraint

$$\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \tag{4.1}$$

according to

$$\begin{aligned} K &:= t(t-1)H \\ &= q(q-1)(q-t)p^2 - [\alpha_4(q-1)(q-t) + \alpha_3 q(q-t) + (\alpha_0 - 1)q(q-1)]\,p \\ &\qquad + \alpha_2(\alpha_1+\alpha_2)(q-t). \end{aligned} \tag{4.2}$$

The time evolution of q and p is governed by the Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}, \tag{4.3}$$

with the relationship to $\mathrm{P_{VI}}$ coming from the fact that eliminating p gives the sixth Painlevé equation in q. A crucial quantity in the development of this viewpoint given in [22] is the τ-function, defined so that

$$H = \frac{d}{dt} \log \tau. \tag{4.4}$$

It was shown in [11] that the $U(N)$ average (1.2) with weight (1.3) is a τ-function, for the $\mathrm{P_{VI}}$ system with the parameters

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \Big(N + 1 + 2\omega_1, N + 2\mu, -N, -\mu - \omega, -\mu - \bar{\omega}\Big), \tag{4.5}$$

and so we can write

$$\begin{aligned} &\tau^{\mathrm{VI}}[N](t; \mu, \omega_1, \omega_2; \xi) \\ &\quad = \Big\langle \prod_{l=1}^{N} (1 - \xi \chi^{(l)}_{(\pi-\phi,\pi)}) e^{\omega_2 \theta_l} |1 + z_l|^{2\omega_1} \left(\frac{1}{t z_l}\right)^{\mu} (1 + t z_l)^{2\mu} \Big\rangle_{U(N)}. \end{aligned} \tag{4.6}$$

The algebraic approach used in [10] makes use of a particular shift operator, or Schlesinger transformation L, constructed from compositions of fundamental reflection operators and Dynkin diagram automorphisms of the $\mathrm{P_{VI}}$ affine Weyl symmetry group $W_a(D_4^{(1)}) = \langle s_0, s_1, s_2, s_3, s_4, r_1, r_3, r_4 \rangle$. Application of the operator L allows for $\tau^{\mathrm{VI}}[N]$ to be computed by a recurrence scheme in N involving auxiliary quantities which satisfy the $\mathrm{dP_V}$ recurrence.

Explicitly, the operator with the property of incrementing N while leaving the other parameters unchanged is $L_{01}^{-1} = r_1 s_0 s_1 s_2 s_3 s_4 s_2$. On the root system parameters α_j in (4.2) it has the action

$$L_{01}^{-1} : \alpha_0 \mapsto \alpha_0 + 1, \alpha_1 \mapsto \alpha_1 + 1, \alpha_2 \mapsto \alpha_2 - 1, \tag{4.7}$$

while α_3 and α_4 remain unchanged. Studying the action of L_{01}^{-1} on $\tau^{\mathrm{VI}}[N]$ in the context of the Okamoto theory, the following result was obtained in [10].

Proposition 4.1 ([10]). *Let $\{g_N, f_N\}_{N=0,1,\ldots}$, satisfy the discrete Painlevé coupled difference equations associated with the degeneration of the rational surface $D_4^{(1)} \to D_5^{(1)}$*

$$g_{N+1} g_N = \frac{t}{t-1} \frac{(f_N + N + 1)(f_N + N + 1 + \mu + \bar{\omega})}{f_N (f_N - \mu - \omega)}, \tag{4.8}$$

$$f_N + f_{N-1} = \mu + \omega + \frac{N + 2\mu}{g_N - 1} + \frac{(N + 1 + 2\omega_1)t}{t(g_N - 1) - g_N}, \tag{4.9}$$

where $t = 1/(1 - e^{i\phi})$ subject to the initial conditions

$$g_0 = \frac{q_0}{q_0 - 1}, \quad f_0 = (1 + \mu + \bar{\omega})(q_0 - 1) + (\mu + \omega) q_0 - (2\omega_1 + 1) \frac{q_0 (q_0 - 1)}{q_0 - t},$$

with

$$q_0 = \frac{1}{2}\left(1 + \frac{i}{\mu}\frac{d}{d\phi}\log e^{i\mu\phi}T_1(e^{i\phi})\right). \tag{4.10}$$

Define $\{q_N, p_N\}_{N=0,1,\dots}$ *by*

$$q_N = \frac{g_N}{g_N - 1},$$

$$\begin{aligned} p_N = {} & \frac{(g_N-1)^2}{g_N} f_N \\ & - (N+1+\mu+\bar{\omega})\frac{g_N-1}{g_N} - (\mu+\omega)(g_N-1) + (N+1+2\omega_1)\frac{g_N-1}{t+(1-t)g_N}. \end{aligned}$$

Then with $T_0(e^{i\phi}) = 1$ *and* $T_1(e^{i\phi}) = w_0(e^{i\phi})$ *as given by (3.1,3.2),* $\{T_N\}_{N=2,3,\dots}$ *is specified by the recurrence*

$$\begin{aligned} & -(N+\mu+\omega)(N+\mu+\bar{\omega})\frac{T_{N+1}T_{N-1}}{T_N^2} \\ & = q_N(q_N-1)p_N^2 + (2\mu+2\omega_1)q_Np_N - (\mu+\bar{\omega})p_N - N(N+2\mu+2\omega_1). \end{aligned} \tag{4.11}$$

An immediate question is the relationship between the Hamiltonian variables q_N, p_N in Proposition 4.1, and the reflection coefficients $r_N, \bar{r}_N$ relating to $\tau^{\mathrm{VI}}[N]$ as studied in Section 3. In fact the quantities are linked by systems of equations given in the following result.

Proposition 4.2. *The transformations linking the Hamiltonian variables* q_N, p_N *in Proposition 4.1 to the reflection coefficients* $r_N, \bar{r}_N$ *for the system of orthogonal polynomials with the weight (1.3) are given implicitly by*

$$\begin{aligned} q_Np_N + \mu + \bar{\omega} = {} & \frac{(N+\mu+\bar{\omega})r_N\bar{r}_N}{(N+\mu+\bar{\omega})r_N\bar{r}_N - \mu + \omega}\frac{1}{q_N-1} \\ & \times\left[(N+2\omega_1)(q_N-1) - t\frac{l_N}{\kappa_N} + Nt + (N+1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{r_{N+1}}{r_N}\right], \end{aligned} \tag{4.12}$$

$$\begin{aligned} & = (N+\mu+\bar{\omega})[(N+\mu+\omega)r_N\bar{r}_N - \mu + \bar{\omega}] \\ & \times\frac{q_N}{(N+2\omega_1)q_N + t\frac{l_N}{\kappa_N} - Nt - (N-1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{\bar{r}_{N-1}}{\bar{r}_N}}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} (q_N-1)p_N + \mu + \omega = {} & (N+\mu+\omega)[(N+\mu+\bar{\omega})r_N\bar{r}_N - \mu + \omega] \\ & \times\frac{q_N-1}{(N+2\omega_1)(q_N-1) - t\frac{l_N}{\kappa_N} + Nt + (N+1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{r_{N+1}}{r_N}}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} & = \frac{(N+\mu+\omega)r_N\bar{r}_N}{(N+\mu+\omega)r_N\bar{r}_N - \mu + \bar{\omega}}\frac{1}{q_N} \\ & \times\left[(N+2\omega_1)q_N + t\frac{l_N}{\kappa_N} - Nt - (N-1+\mu+\bar{\omega})(1-r_N\bar{r}_N)t\frac{\bar{r}_{N-1}}{\bar{r}_N}\right]. \end{aligned} \tag{4.15}$$

Proof. We require in addition to the primary shift operator L_{01}^{-1} generating the $N \mapsto N+1$ sequence another operator which has the action $i\omega_2 \mapsto i\omega_2 - 1$. This is the secondary shift operator $T_{34}^{-1} = r_1 s_4 s_2 s_0 s_1 s_2 s_4$ and has the action $T_{34}^{-1} : \alpha_3 \to \alpha_3 + 1, \alpha_4 \to \alpha_4 - 1$. From Table 1 of [11] we compute the actions of T_{34}^{-1}, T_{34} on the Hamiltonian to be

$$\begin{aligned} T_{34}^{-1} \cdot K_n - K_n = &-q_n(q_n-1)p_n \\ &+ (\alpha_0+\alpha_4-1)(q_n-1) - (\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)\frac{q_n-1}{(q_n-1)p_n-\alpha_3}, \end{aligned}$$

$$\begin{aligned} T_{34} \cdot K_n - K_n = &-q_n(q_n-1)p_n \\ &+ (\alpha_0+\alpha_3-1)q_n - (\alpha_2+\alpha_4)(\alpha_1+\alpha_2+\alpha_4)\frac{q_n}{q_n p_n-\alpha_4}. \end{aligned}$$

However

$$T_{34}^{-1} \cdot K_n - K_n = t(t-1)\frac{d}{dt}\log\frac{I_n^1}{I_n^0} = t(t-1)\frac{d}{dt}\log r_n,$$

and we employ the results of 2.30 and the evaluation of the coefficient functions in (3.7,3.8) to arrive at

$$\begin{aligned} (t-1)\frac{\dot{r}_N}{r_N} &= \frac{l_N}{\kappa_N} - N - (N+1+\mu+\bar{\omega})(1-r_N\bar{r}_N)\frac{r_{N+1}}{r_N}, \\ (t-1)\frac{\dot{\bar{r}}_N}{\bar{r}_N} &= -\frac{l_N}{\kappa_N} + N + (N-1+\mu+\bar{\omega})(1-r_N\bar{r}_N)\frac{\bar{r}_{N-1}}{\bar{r}_N}. \end{aligned}$$

In addition we note that after recalling (2.6), (4.11) factorises into

$$(N+\mu+\omega)(N+\mu+\bar{\omega})r_N\bar{r}_N = [q_N p_N + \mu + \bar{\omega}][(q_N-1)p_N + \mu + \omega].$$

The stated results, (4.12-4.15), then follow. □

There is another perspective on $\tau^{\rm VI}[N]$ for which the Okamoto τ-function theory can be used to provide a recurrence system based on $\rm dP_V$ distinct from that in Proposition 4.1. The starting point, used extensively in [11], is to obtain $\tau^{\rm VI}[N]|_{\xi=0}$ as specified by 4.6 via the projection $(-1,1) \to \mathbb{T}$ of an average over the Jacobi unitary ensemble. This relates (4.6) to the $\rm P_{VI}$ system with the parameters

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \Big(1-\mu-\omega, N+2\mu, -N, -\mu-\bar{\omega}, N+2\omega_1\Big). \tag{4.16}$$

Sequences of the Hamiltonian variables $\{q_n, p_n, H_n, \tau_n\}_{n=0,1,\dots}$ are now generated by the shift operator $L_{14}^{-1} = r_3 s_1 s_4 s_2 s_0 s_3 s_2$. It has the action $L_{14}^{-1} : \alpha_1 \mapsto \alpha_1 + 1, \alpha_2 \mapsto \alpha_2 - 1, \alpha_4 \mapsto \alpha_4 + 1$. Using the methods of [11] we have the following result.

Lemma 4.1. *The sequence of auxiliary variables* $\{g_n, f_n\}_{n=0,1,\dots}$ *defined by*

$$g_n := \frac{q_n - t}{q_n - 1}, \tag{4.17}$$

$$f_n := \frac{1}{1-t}\Big[(q_n-t)(q_n-1)p_n + (1-\alpha_0-\alpha_2)(q_n-1) - \alpha_3(q_n-t) - \alpha_4\frac{(q_n-t)(q_n-1)}{q_n}\Big], \tag{4.18}$$

generated by the shift operator L_{14}^{-1} satisfies the discrete Painlevé equations (1.6), (1.7).

Proof. Using the action of the fundamental reflections and Dynkin diagram automorphisms given in Table 1 of [11] we compute the action of L_{14}^{-1} on q and write it in the following way,

$$\begin{aligned}\frac{(q-t)(\hat{q}-t)}{(q-1)(\hat{q}-1)} &= t[q(q-1)(q-t)p+(\alpha_1+\alpha_2)q^2-((\alpha_0+\alpha_1+\alpha_2)t-\alpha_0-\alpha_4)q-\alpha_4 t]\\ &\times[q(q-1)(q-t)p+(\alpha_1+\alpha_2)q^2-((\alpha_1+\alpha_2)t-\alpha_4)q-\alpha_4 t]\\ &\div[q(q-1)(q-t)p+(\alpha_1+\alpha_2)q^2-(-\alpha_4 t+\alpha_1+\alpha_2)q-\alpha_4 t]\\ &\div[q(q-1)(q-t)p+(\alpha_1+\alpha_2)q^2-(-(\alpha_3+\alpha_4)t+\alpha_1+\alpha_2+\alpha_3)q-\alpha_4 t],\end{aligned}$$

where $q := q_n, \hat{q} := q_{n+1}$. From the definitions (4.17,4.18) this result can be readily recast as (1.6). The second (1.7) follows from a computation for $f_n + f_{n-1}$ using the shift operator L_{14}. □

Making use of Lemma 4.1, together with elements of the Okamoto theory as detailed in [11], and applying this to derive recurrences according to the strategy of [10], gives the following recurrence scheme for $\{\tau^{\rm VI}[N]\}_{N=0,1,2,\dots}$.

Proposition 4.3. *Let $\{g_N, f_N\}_{N=0,1,\dots}$, satisfy the* $\mathrm{dP_V}$ *system*

$$g_{N+1}g_N = t\frac{(f_N+N+1)(f_N+N+\mu+\omega)}{f_N(f_N-\mu-\bar{\omega})}, \tag{4.19}$$

$$f_N+f_{N-1} = \mu+\bar{\omega}+\frac{N+2\mu}{g_N-1}+\frac{(N+2\omega_1)t}{g_N-t}, \tag{4.20}$$

where $t=e^{i\phi}$ subject to the initial conditions

$$g_0 = \frac{q_0-t}{q_0-1},$$

$$f_0 = \frac{1}{1-t}\left[(\mu+\omega)(q_0-1)+(\mu+\bar{\omega})(q_0-t)-2\omega_1\frac{(q_0-t)(q_0-1)}{q_0}\right],$$

with

$$q_0 = \frac{\omega_1}{\mu}\frac{-i\frac{d}{d\phi}\log e^{i\mu\phi}T_1(e^{i\phi})}{\mu+\omega+i\frac{d}{d\phi}\log e^{i\mu\phi}T_1(e^{i\phi})}. \tag{4.21}$$

Define $\{q_N, p_N\}_{N=0,1,\dots}$ *in terms of* $\{f_N, g_N\}_{N=0,1,\dots}$ *by*

$$q_N = \frac{g_N - t}{g_N - 1}, \tag{4.22}$$

$$p_N = \frac{g_N - 1}{(1-t)g_N}\Big[(g_N - 1)f_N - (\mu + \bar{\omega})g_N + (N + 2\omega_1)\frac{(1-t)g_N}{g_N - t} - N - \mu - \omega\Big]. \tag{4.23}$$

Then with $T_0(e^{i\phi}) = 1$ *and* $T_1(e^{i\phi}) = w_0(e^{i\phi})$ *as given by (3.1,3.2),* $\{T_N\}_{N=2,3,\dots}$ *is specified by the recurrence*

$$\begin{aligned} &-(N+\mu+\omega)(N+\mu+\bar{\omega})\frac{T_{N+1}T_{N-1}}{T_N^2} \\ &= q_N(q_N-1)^2p_N^2 + [(2\mu - N)q_N + N + 2\omega_1](q_N - 1)p_N - 2\mu N q_N - N(N+2\omega_1). \end{aligned} \tag{4.24}$$

Proof. Let $Y_n := L_{14}^{-1}K_n - K_n = K_{n+1} - K_n$. From Table 1 of [11] we have

$$Y_n = -\frac{(t-1)q_n}{q_n - 1}\left\{(q_n - 1)p_n + \alpha_0 + \alpha_2 - 1 + \frac{(1 - \alpha_0 - \alpha_2)(\alpha_1 + \alpha_2 + \alpha_4)}{q_n(q_n-1)p_n + (\alpha_1 + \alpha_2)q_n + \alpha_4}\right\}.$$

Now consider

$$\begin{aligned} t(t-1)\frac{d}{dt}\log\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} &= K_{n+1} + K_{n-1} - 2K_n, \\ &= Y_n - L_{14}Y_n. \end{aligned}$$

This latter difference, upon again consulting Table 1 of [11], turns out to be

$$\begin{aligned} Y_n - L_{14}Y_n = t(t-1)\frac{d}{dt}\log\Big(&q_n(q_n-1)^2p_n^2 \\ &+ [(\alpha_1 + 2\alpha_2)q_n + \alpha_4](q_n - 1)p_n + \alpha_2[(\alpha_1 + \alpha_2)q_n + \alpha_4]\Big). \end{aligned}$$

After integrating both expressions and introducing an integration constant (4.24) follows. □

Remark 4.1. It is known from [10],[11] that the sequence of auxiliary variables $\{g_n, f_n\}_{n=0,1,\dots}$ defined by

$$g_n := \frac{q_n}{q_n - 1}, \tag{4.25}$$

$$f_n := q_n(q_n-1)p_n + (1 - \alpha_2 - \alpha_4)(q_n - 1) - \alpha_3 q_n - \alpha_0\frac{q_n(q_n-1)}{q_n - t}, \tag{4.26}$$

generated by the shift operator L_{01}^{-1} satisfy the $\mathrm{dP_V}$ equations

$$g_{n+1}g_n = \frac{t}{t-1}\frac{(f_n + 1 - \alpha_2)(f_n + 1 - \alpha_2 - \alpha_4)}{f_n(f_n + \alpha_3)}, \tag{4.27}$$

$$f_n + f_{n-1} = -\alpha_3 + \frac{\alpha_1}{g_n - 1} + \frac{\alpha_0 t}{t(g_n - 1) - g_n}. \tag{4.28}$$

In fact the two systems of recurrences (4.27,4.28) and (1.6,1.7) are related by an element of the S_4 subgroup of the $W_a(F_4)$ transformations, namely the generator x^3 [22]. This has the action

$$x^3 : \alpha_0 \leftrightarrow \alpha_4, t \mapsto \frac{t}{t-1}, q \mapsto \frac{t-q}{t-1}, p \mapsto -(t-1)p, \tag{4.29}$$

and when applying these transformations to (4.25), (4.26), (4.27), (4.28) we recover (4.17), (4.18), (1.6), (1.7) respectively.

5. Applications to Physical Models

5.1. **Random Matrix Averages.** A specialisation of the above results with great interest in the application of random matrices [18],[17] is the quantity

$$F_N^{\mathrm{CUE}}(u;\mu) := \Big\langle \prod_{l=1}^{N} |u+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N}. \tag{5.1}$$

This has the interpretation as the average of the 2μ-th power, or equivalently the 2μ-th moment, of the absolute value of the characteristic polynomial for the CUE. In the case $|u|=1$ (5.1) is independent of u and has the well-known (see e.g. [3]) gamma function evaluation

$$\Big\langle \prod_{l=1}^{N} |u+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N}\Big|_{u=e^{i\phi}} = \Big\langle \prod_{l=1}^{N} |1+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N} = \prod_{j=0}^{N-1} \frac{j!\Gamma(j+1+2\mu)}{\Gamma^2(j+1+\mu)}, \tag{5.2}$$

where for convergence of the integral $\Re(\mu) > -1/2$. For $|u|<1$ we see by an appropriate change of variables that

$$\begin{aligned} \Big\langle \prod_{l=1}^{N} |u+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N} &= \Big\langle \prod_{l=1}^{N} (1+|u|^2 z_l)^{\mu}(1+1/z_l)^{\mu} \Big\rangle_{\mathrm{CUE}_N}, \\ &= {}_2F_1^{(1)}(-\mu,-\mu;N;t_1,\dots,t_N)|_{t_1=\dots=t_N=|u|^2}, \end{aligned} \tag{5.3}$$

where the second equality follows from (3.50). For $|u|>1$ we can use the simple functional equation

$$\Big\langle \prod_{l=1}^{N} |u+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N} = |u|^{2\mu N} \Big\langle \prod_{l=1}^{N} |\frac{1}{u}+z_l|^{2\mu} \Big\rangle_{\mathrm{CUE}_N}, \tag{5.4}$$

to relate this case back to the case $|u|<1$.

The weight in the first equality of (5.3) is a special case of (1.3). In terms of the parameters of the form (1.3) we observe that $\xi=0$, $2\mu \mapsto \mu$, $\omega=\bar{\omega}=\mu/2$, i.e. $\omega_2=0$ and $t=|u|^2$. The trigonometric moments are

$$w_{-n} = \frac{\Gamma(\mu+1)}{n!\Gamma(\mu+1-n)}\, {}_2F_1(-\mu,-\mu+n;n+1;|u|^2) \qquad n \in \mathbb{Z}_{\geq 0}, \tag{5.5}$$

$$w_n = |u|^{2n} w_{-n} \qquad n \in \mathbb{Z}_{\geq 0}. \tag{5.6}$$

The results of Section 3 then allow (5.1) to be computed by a recurrence involving the corresponding reflection coefficients.

Corollary 5.1. *The general moments of the characteristic polynomial* $|\det(u+U)|$ *for arbitrary exponent* 2μ *with respect to the finite CUE ensemble* $U \in U(N)$ *of rank* N *is given by the system of recurrences*

$$\frac{F^{\rm CUE}_{N+1}F^{\rm CUE}_{N-1}}{(F^{\rm CUE}_{N})^2} = 1 - |u|^{2N} r_N^2, \tag{5.7}$$

with initial values

$$F^{\rm CUE}_0 = 1, \qquad F^{\rm CUE}_1 = {}_2F_1(-\mu, -\mu; 1; |u|^2), \tag{5.8}$$

and

$$\begin{aligned} 2|u|^{2N} r_N r_{N-1} - |u|^2 - 1 &= \frac{1 - |u|^{2N} r_N^2}{r_N}\left[(N+1+\mu)|u|^2 r_{N+1} + (N-1+\mu) r_{N-1}\right] \\ &\quad - \frac{1 - |u|^{2(N-1)} r_{N-1}^2}{r_{N-1}}\left[(N+\mu)|u|^2 r_N + (N-2+\mu) r_{N-2}\right], \end{aligned} \tag{5.9}$$

subject to the initial values

$$r_0 = 1, \quad r_1 = -\mu \frac{{}_2F_1(-\mu, -\mu+1; 2; |u|^2)}{{}_2F_1(-\mu, -\mu; 1; |u|^2)}. \tag{5.10}$$

Proof. From either (3.9), (3.10) or (3.33) and the fact that $\bar{r}_1 = |u|^2 r_1$ we can repeat the arguments of Corollary 3.4 to deduce that $\bar{r}_N = |u|^{2N} r_N$ for $N \geq 0$. The recurrence relation follows simply from the specialisation of (3.19) and the initial conditions from the $N = 1$ case. □

Another spectral statistic of fundamental importance in random matrix theory is the generating function $E^{\rm CUE}_N((\pi-\phi,\pi);\xi)$ for the probabilities $E^{\rm CUE}_N(k;(\pi-\phi,\pi))$ that exactly k eigenvalues lie in the interval $(\pi-\phi,\pi)$. This is specified by

$$\begin{aligned} &E^{\rm CUE}_N((\pi-\phi,\pi);\xi) \\ &:= \frac{1}{C_N}\left(\int_{-\pi}^{\pi} - \xi\int_{\pi-\phi}^{\pi}\right) d\theta_1 \ldots \left(\int_{-\pi}^{\pi} - \xi\int_{\pi-\phi}^{\pi}\right) d\theta_N \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2, \end{aligned} \tag{5.11}$$

where $C_N = (2\pi)^N N!$, and thus corresponds to the special case $\mu = \omega = \bar{\omega} = 0$, $t = e^{i\phi}$ of (1.3). We remark that for these parameters the underlying Toeplitz matrix elements, given for general parameters by (3.1) or (3.2), have the elementary form

$$w_n = \delta_{n,0} + \frac{\xi}{2\pi i}(-1)^{n+1}\frac{t^n - 1}{n}, \tag{5.12}$$

A recurrence scheme for the generating function $E^{\rm CUE}_N$, deduced as a corollary of Proposition 4.1, has been presented in [10]. Here we use recurrences found herein for r_N, $\bar{r}_N$, together with the fact that for (5.11) one has $r_N = t^{-N}\bar{r}_N$, to replace the role of the coupled recurrences from [10] by a single recurrence.

Corollary 5.2. *The generating function for the probability of finding exactly k eigenvalues $z = e^{i\theta}$ from the ensemble of random $N \times N$ unitary matrices within the sector of the unit circle $\theta \in (\pi - \phi, \pi]$ is given by the following system of recurrences in the rank of the ensemble N,*

$$\frac{E^{\rm CUE}_{N+1} E^{\rm CUE}_{N-1}}{(E^{\rm CUE}_{N})^2} = 1 - x_N^2, \tag{5.13}$$

where the initial values are

$$E^{\rm CUE}_0 = 1, \quad E^{\rm CUE}_1 = 1 - \frac{\xi}{2\pi}\phi, \tag{5.14}$$

and the auxiliary variables x_N are determined by the quasi-linear third order recurrence relation

$$\begin{aligned} 2x_N x_{N-1} - 2\cos\frac{\phi}{2} = \frac{1-x_N^2}{x_N}\left[(N+1)x_{N+1} + (N-1)x_{N-1}\right] \\ - \frac{1-x_{N-1}^2}{x_{N-1}}\left[N x_N + (N-2)x_{N-2}\right], \end{aligned} \tag{5.15}$$

or the quadratic second order recurrence relation

$$\begin{aligned} (1-x_N^2)^2\left[(N+1)^2 x_{N+1}^2 + (N-1)^2 x_{N-1}^2\right] + 2(N^2-1)(1-x_N^4)x_{N+1}x_{N-1} \\ + 4N\cos\frac{\phi}{2} x_N(1-x_N^2)\left[(N+1)x_{N+1} + (N-1)x_{N-1}\right] \\ + 4N^2 x_N^2\left[\cos^2\frac{\phi}{2} - x_N^2\right] = 0, \end{aligned} \tag{5.16}$$

along with the initial values

$$x_{-1} = 0, \quad x_0 = 1, \quad x_1 = -\frac{\xi}{\pi}\frac{\sin\frac{\phi}{2}}{1 - \frac{\xi}{2\pi}\phi}. \tag{5.17}$$

Proof. The first recurrence relation follows directly from the general recurrence (3.19) and Corollary 3.4 whilst the second follows from (3.27). □

5.2. **2-D Ising Model.** Consider the two-dimensional Ising model with dimensionless nearest neighbour couplings equal to K_1 and K_2 in the x and y directions respectively (see e.g. [4]). Let $\sigma_{0,0}$ and $\sigma_{N,N}$ denote the values of the spins at the lattice sites $(0,0)$ and (N,N) respectively. For the infinite lattice an unpublished result of Onsager (see [21]) gives that the diagonal spin-spin correlation $\langle\sigma_{0,0}\sigma_{N,N}\rangle$ has the Toeplitz form

$$\langle\sigma_{0,0}\sigma_{N,N}\rangle = \det(a_{i-j}(k))_{1\le i,j\le N}, \tag{5.18}$$

where

$$a_p(k) := \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-ip\theta}\left[\frac{1-(1/k)e^{-i\theta}}{1-(1/k)e^{i\theta}}\right]^{1/2}, \tag{5.19}$$

$k = \sinh 2K_1 \sinh 2K_2$. The analytic structure is different depending on $k > 1$ (low temperature phase) or $k < 1$ (high temperature phase).

Jimbo and Miwa [15] identified (5.18) as the τ-function associated with a monodromy preserving deformation of a linear system, which in turn is associated with $\mathrm{P_{VI}}$. In a more recent work [11] the present authors have identified (5.18) as a τ-function in the Okamoto theory of $\mathrm{P_{VI}}$ [22]. These identifications have the consequence of allowing (5.18) to be characterised in terms of a solution of the σ-form of the Painlevé VI equation, when regarded as a function of $t := k^2$.

It is our objective in this subsection to use the recurrences of Subsection 3.1, appropriately specialised, to derive a N-recurrence for (5.18). Before doing so, we remark that a recent result of Borodin [6] can also be used for the same purpose. The latter recurrence applies to all Toeplitz determinants

$$q_n^{(z,z',\xi)} := (1-\xi)^{zz'} \det[g_{i-j}]_{i,j=1,\dots,n}, \tag{5.20}$$

where

$$g_p = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-ip\theta}\left(1-\sqrt{\xi}e^{i\theta}\right)^z\left(1-\sqrt{\xi}e^{-i\theta}\right)^{z'}. \tag{5.21}$$

As previously noted in [11], comparison with (5.18) and (5.19) shows

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-k^{-2})^{1/4} q_N^{(-1/2,1/2,1/k^2)}. \tag{5.22}$$

By regarding the Fourier integral in (5.19) as a contour integral, and changing the contour of integration, the corresponding weight function can be chosen to be equal to [11]

$$z^{1/4}|1+z|^{-1/2}(1+k^{-2}z)^{1/2} = z^{1/2}(1+z)^{-1/2}(1+k^{-2}z)^{1/2}, \quad 1/k^2 < 1, \tag{5.23}$$

$$k^{-1}z^{-3/4}|1+z|^{-1/2}(1+k^2z)^{1/2} = k^{-1}z^{-1/2}(1+z)^{-1/2}(1+k^2z)^{1/2}, \quad k^2 < 1. \tag{5.24}$$

The first, multiplied by $k^{1/2}$, is the case

$$\xi = 0, \mu = \frac{1}{4}, \omega_1 = -\frac{1}{4}, \omega_2 = \frac{i}{2}, t = 1/k^2, \tag{5.25}$$

of (1.3), while the second multiplied by $k^{1/2}$ is the case

$$\xi = 0, \mu = \frac{1}{4}, \omega_1 = -\frac{1}{4}, \omega_2 = -\frac{i}{2}, t = k^2. \tag{5.26}$$

Substituting these values in (3.1) gives Gauss hypergeometric evaluations for the matrix elements (5.19). After an appropriate limiting procedure to account for factors which otherwise would be singular, one obtains the well known fact that the Toeplitz elements in the low temperature regime are given by

$$w_{-n} = \frac{(-1)^n}{\pi}\frac{\Gamma(n+1/2)\Gamma(1/2)}{\Gamma(n+1)}\, {}_2F_1(-1/2, n+1/2; n+1; k^{-2}), \quad n \geq 0, \tag{5.27}$$

$$w_n = \frac{(-1)^{n+1}k^{-2n}}{\pi}\frac{\Gamma(n-1/2)\Gamma(3/2)}{\Gamma(n+1)}\, {}_2F_1(1/2, n-1/2; n+1; k^{-2}), \quad n > 0, \tag{5.28}$$

whilst those in the high temperature regime are

$$w_{-n} = \frac{(-1)^n k^{2n+1}}{\pi} \frac{\Gamma(n+1/2)\Gamma(3/2)}{\Gamma(n+2)} {}_2F_1(1/2, n+1/2; n+2; k^2), \quad n \geq 0, \tag{5.29}$$

$$w_n = \frac{(-1)^{n-1}}{\pi k} \frac{\Gamma(n-1/2)\Gamma(1/2)}{\Gamma(n)} {}_2F_1(-1/2, n-1/2; n; k^2), \quad n > 0. \tag{5.30}$$

The results of Subsection 3.1 provide the following recurrence scheme.

Corollary 5.3. *The diagonal correlation function for the Ising model valid in both the low and high temperature phases (with $k \mapsto 1/k$ in the latter case) is determined by*

$$\frac{\langle \sigma_{0,0}\sigma_{N+1,N+1}\rangle \langle \sigma_{0,0}\sigma_{N-1,N-1}\rangle}{\langle \sigma_{0,0}\sigma_{N,N}\rangle^2} = 1 - r_N \bar{r}_N, \tag{5.31}$$

along with the quasi-linear $2/1$

$$\begin{aligned}(2N+3)k^{-2}(1-r_N\bar{r}_N)r_{N+1} + 2N\left[k^{-2} + 1 - (2N-1)k^{-2}r_N\bar{r}_{N-1}\right] r_N \\ + (2N-3)\left[(2N-1)r_N\bar{r}_N + 1\right] r_{N-1} = 0,\end{aligned} \tag{5.32}$$

and $1/2$ *recurrence relations*

$$\begin{aligned}(2N+1)(1-r_N\bar{r}_N)\bar{r}_{N+1} + 2N\left[(2N-3)\bar{r}_N r_{N-1} + k^{-2} + 1\right] \bar{r}_N \\ + (2N-1)k^{-2}\left[-(2N+1)r_N\bar{r}_N + 1\right] \bar{r}_{N-1} = 0,\end{aligned} \tag{5.33}$$

subject to initial conditions for the low temperature regime

$$r_0 = \bar{r}_0 = 1,\ r_1 = \frac{2-k^2}{3} + \frac{k^2-1}{3}\frac{\mathrm{K}(k^{-1})}{\mathrm{E}(k^{-1})},\ \bar{r}_1 = -1 + \frac{k^2-1}{k^2}\frac{\mathrm{K}(k^{-1})}{\mathrm{E}(k^{-1})}, \tag{5.34}$$

or to the initial conditions for the high temperature regime given by

$$r_0 = \bar{r}_0 = 1,\ r_1 = \frac{1}{3}\left\{\frac{2}{k^2} - \frac{\mathrm{E}(k)}{(k^2-1)\mathrm{K}(k) + \mathrm{E}(k)}\right\},\ \bar{r}_1 = -\frac{k^2\mathrm{E}(k)}{(k^2-1)\mathrm{K}(k) + \mathrm{E}(k)}, \tag{5.35}$$

where $\mathrm{K}(k), \mathrm{E}(k)$ *are the complete elliptic integrals of the first and second kind respectively.*

Proof. The equations (5.32,5.33) follow from (3.33) and its "conjugate" upon specialisation of the parameters as required by (5.25) and (5.26). The initial conditions follow from (3.21), (5.27)-(5.30) and the relationship between the Gauss hypergeometric functions therein and the complete elliptic integrals. □

A consequence of the results of Subsection 3.3 is the following result relating the quantities of Corollary 5.3 to the generalised hypergeometric function ${}_2F_1^{(1)}$.

Corollary 5.4. *In the low temperature phase the diagonal correlation function is given by*

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = {}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \ldots, t_N)|_{t_1=\ldots=t_N=1/k^2}, \tag{5.36}$$

whilst the reflection coefficients are given by

(5.37)

$$r_N = (-1)^N \frac{(-1/2)_N}{N!} \frac{{}_2F_1^{(1)}(-1/2, 3/2; N+1; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \ldots, t_N)}\bigg|_{t_1=\ldots=t_N=1/k^2},$$

(5.38)

$$\bar{r}_N = (-1)^N \frac{(N-1)!}{(1/2)_N} \frac{\lim_{\epsilon\to 0} \epsilon\, {}_2F_1^{(1)}(-1/2, -1/2; N-1+\epsilon; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(-1/2, 1/2; N; t_1, \ldots, t_N)}\bigg|_{t_1=\ldots=t_N=1/k^2}.$$

In the high temperature phase the diagonal correlation function is

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = \frac{(2N-1)!!}{2^N N!} k^N {}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \ldots, t_N)|_{t_1=\ldots=t_N=k^2}, \tag{5.39}$$

and the reflection coefficients are given by

$$r_N = (-1)^N \frac{(-1/2)_N}{(N+1)!} \frac{{}_2F_1^{(1)}(1/2, 3/2; N+2; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \ldots, t_N)}\bigg|_{t_1=\ldots=t_N=k^2}, \tag{5.40}$$

$$\bar{r}_N = (-1)^N \frac{N!}{(1/2)_N} \frac{{}_2F_1^{(1)}(1/2, -1/2; N; t_1, \ldots, t_N)}{{}_2F_1^{(1)}(1/2, 1/2; N+1; t_1, \ldots, t_N)}\bigg|_{t_1=\ldots=t_N=k^2}. \tag{5.41}$$

Proof. The evaluations (5.36) and (5.39) follow by specialising the parameters in (3.50) as required by (5.25) and (5.26) respectively. The evaluations (5.37), (5.38) in the low temperature phase follow from (3.51), (3.52) by specialising the parameters as required by (5.25). Some care needs to be taken with $\bar{r}_N$ because $-\mu + \bar{\omega} = 0$. Applying a limiting process leads to

$$\begin{aligned} \lim_{\epsilon\to 0} \epsilon\, {}_2F_1^{(1)}&(-1/2, -1/2; N-1+\epsilon; t_1, \ldots, t_N) \\ &= \sum_{\kappa : l(\kappa)=N} \frac{([-1/2]_\kappa^{(1)})^2}{[N]_\kappa^{(1)}} \frac{\prod_{j=1}^N (N-j+\kappa_j)}{(N-1)!} \frac{s_\kappa(t_1, \ldots, t_N)}{h_\kappa}, \end{aligned} \tag{5.42}$$

in which only those terms with lengths $l(\kappa) = N$ contribute to the sum. The high temperature expressions follow from the low temperature ones through the transformation $\mu \leftrightarrow \omega_1$. □

It is of interest to note that as N grows more of the leading order terms in the expansion of (5.36) become independent of N, and the following limit becomes explicit

$$\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-k^{-2})^{1/4}. \tag{5.43}$$

At zero temperature, $k = \infty$, the solutions simplify to

$$r_N = (-1)^N \frac{(-1/2)_N}{N!}, \quad \bar{r}_N = 0 \; (N \geq 1), \quad \langle \sigma_{0,0}\sigma_{N,N}\rangle = 1, \tag{5.44}$$

whilst at the critical point, $k = 1$, we have the simple solutions

$$r_N = \frac{(-1)^{N-1}}{(2N+1)(2N-1)}, \quad \bar{r}_N = (-1)^N, \quad l_N = \frac{N}{2N+1}, \tag{5.45}$$

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = \prod_{j=1}^{N} \frac{\Gamma^2(j)}{\Gamma(j+{}^1\!/\!_2)\Gamma(j-{}^1\!/\!_2)}. \tag{5.46}$$

At infinite temperature they become

$$r_N = (-1)^N \frac{(-{}^1\!/\!_2)_N}{(N+1)!}, \quad \bar{r}_N = (-1)^N \frac{N!}{({}^1\!/\!_2)_N}, \quad \langle \sigma_{0,0}\sigma_{N,N} \rangle = 0 \ (N \geq 1). \tag{5.47}$$

These are all in agreement with the known results [21].

Acknowledgments

This research has been supported by the Australian Research Council. NSW appreciates the generosity of Will Orrick in supplying expansions of the Toeplitz determinants for the diagonal correlations of the Ising model and the assistance of Paul Leopardi in calculating gap probabilities for the CUE.

References

1. M. Adler and P. van Moerbeke, *Recursion relations for unitary integrals, combinatorics and the Toeplitz lattice*, Comm. Math. Phys. **237** (2003), no. 3, 397–440. MR 1 993 333
2. J. Baik and E. M. Rains, *Algebraic aspects of increasing subsequences*, Duke Math. J. **109** (2001), no. 1, 1–65. MR 1 844 203
3. T. H. Baker and P. J. Forrester, *The Calogero-Sutherland model and generalized classical polynomials*, Comm. Math. Phys. **188** (1997), no. 1, 175–216. MR **99c:**33012
4. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.
5. A. Borodin and G. Olshanski, *Distributions on partitions, point processes, and the hypergeometric kernel*, Comm. Math. Phys. **211** (2000), no. 2, 335–358. MR **2001k:**33031
6. Alexei Borodin, *Discrete gap probabilities and discrete Painlevé equations*, Duke Math. J. **117** (2003), no. 3, 489–542. MR 1 979 052
7. Alexei Borodin and Dmitriy Boyarchenko, *Distribution of the first particle in discrete orthogonal polynomial ensembles*, Comm. Math. Phys. **234** (2003), no. 2, 287–338. MR 1 962 463
8. P. J. Forrester, *Selberg correlation integrals and the $1/r^2$ quantum many-body system*, Nuclear Phys. B **388** (1992), no. 3, 671–699. MR **MR1201273 (94e:**33030)
9. P. J. Forrester, N. E. Frankel, T. Garoni, and N. S. Witte, *Painlevé transcendent evaluations of finite system density matrices for 1d impenetrable bosons*, Comm. Math. Phys. **238** (2003), no. 1-2, 257–285.
10. P. J. Forrester and N. S. Witte, *Discrete Painlevé equations and random matrix averages*, Nonl. **16** (2003), no. 6, 1919–1944.
11. ——, *Application of the τ-function theory of Painlevé equations to random matrices: PVI, the JUE, CyUE, cJUE and scaled limits*, Nagoya Math. J. **174** (2004), 29–114.
12. ——, *Bi-orthogonal Polynomials on the Unit Circle, regular semi-classical Weights and Integrable Systems*, 2004.

13. ———, *Discrete Painlevé Equations, Orthogonal Polynomials on the Unit Circle and n-recurrences for averages over* $U(N)$ – $\mathrm{P}_{\mathrm{III}'}$ *and* P_{V} τ*-functions*, Int. Math. Res. Not. **2004** (2004), no. 4, 159–183.

14. Géza Freud, *On the coefficients in the recursion formulae of orthogonal polynomials*, Proc. Roy. Irish Acad. Sect. A **76** (1976), no. 1, 1–6. MR 54 #7913

15. M. Jimbo and T. Miwa, *Studies on holonomic quantum fields. XVII*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 9, 405–410. MR **85h:**82016

16. J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math. Anal. **24** (1993), no. 4, 1086–1110. MR **94h:**33010

17. J. P. Keating and F. Mezzadri, *Random Matrix Theory and Entanglement in Quantum Spin Chains*, arXiv:quant-ph/0407047.

18. J. P. Keating and N. C. Snaith, *Random matrix theory and* $\zeta(1/2+it)$, Comm. Math. Phys. **214** (2000), no. 1, 57–89. MR **2002c:**11107

19. F. Klein, *Vorlesungen über die Hypergeometrische Function*, Verlag von Julius Springer, Inc., Berlin, 1933, Gehalten an der Universität Göttingen in Wintersemester 1893/94.

20. A. P. Magnus, *MAPA3072A Special topics in approximation theory 1999-2000: Semi-classical orthogonal polynomials on the unit circle*, http://www.math.ucl.ac.be/~magnus/.

21. B. McCoy and T. T. Wu, *The Two-Dimensional Ising Model*, Harvard University Press, Harvard, 1973.

22. K. Okamoto, *Studies on the Painlevé equations. I. Sixth Painlevé equation* P_{VI}, Ann. Mat. Pura Appl. (4) **146** (1987), 337–381. MR **88m:**58062

23. H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Comm. Math. Phys. **220** (2001), no. 1, 165–229. MR 1 882 403

24. Z. M. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math. **44** (1992), no. 6, 1317–1338. MR **94c:**33026

Department of Mathematics and Statistics, University of Melbourne, Victoria 3010, Australia

E-mail address: p.forrester@ms.unimelb.edu.au; n.witte@ms.unimelb.edu.au